

A Pohozaev-type formula and Quantization of Horizontal Half-Harmonic Maps

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Abstract

In a recent paper [9] the first and the third authors introduced the notion of *horizontal α harmonic map*, $\alpha \geq 1/2$ with respect to a given C^1 planes distribution P_T on all \mathbb{R}^m . These are maps $u \in \dot{H}^\alpha(\mathbb{R}^k, \mathbb{R}^m)$, $\alpha \geq 1/2$, satisfying $P_T \nabla u = \nabla u$ and $P_T(u)(-\Delta)^\alpha u = 0$ in $\mathcal{D}'(\mathbb{R}^k)$. The goal of this paper is to investigate compactness and quantization properties of sequences of horizontal $1/2$ harmonic maps u_k in 1D such that $\|u_k\|_{\dot{H}^{1/2}(\mathbb{R})} \leq C$ and $\|(-\Delta)^{1/2} u_k\|_{L^1(\mathbb{R})} \leq C$. We show that there exist a horizontal $1/2$ harmonic map u_∞ and a possibly empty set $\{a_1, \dots, a_\ell\}$, $\ell \geq 1$, such that up to subsequence

$$u_k \rightarrow u_\infty \quad \text{in } \dot{W}_{loc}^{1/2,p}(\mathbb{R} \setminus \{a_1, \dots, a_\ell\}), \quad p \geq 2 \text{ as } k \rightarrow +\infty. \quad (1)$$

Moreover there is a family $\tilde{u}_\infty^{i,j}$ of horizontal $1/2$ -harmonic maps ($i \in \{1, \dots, \ell\}, j \in \{1, \dots, N_i\}$), such that up to subsequence

$$\left\| (-\Delta)^{1/4} \left(u_k - u_\infty - \sum_{i,j} \tilde{u}_\infty^{i,j}((x - x_{i,j}^k)/r_{i,j}^k) \right) \right\|_{L_{loc}^2(\mathbb{R})} \rightarrow 0, \quad \text{as } k \rightarrow +\infty$$

for some sequences $r_{i,j}^k \rightarrow 0$ and $x_{i,j}^k \rightarrow a_i$ as $k \rightarrow \infty$.

The quantization analysis is obtained through a precise asymptotic development of the energy of u_k in the neck region and a subtle application of new Pohozaev-type formulae.

Key words. Horizontal fractional harmonic map, Schrödinger-type PDEs, conservation laws, regularity of solutions, blow-up analysis.

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Contents

1	Introduction	2
1.1	The strategy of the proof of theorem 1.2.	8
1.2	Definitions and Notations	13
2	Preliminary results on nonlocal Schrödinger system	14
3	Pohozaev Identities	19
3.1	Pohozaev Identities for $(-\Delta)^{1/2}$ in \mathbb{R}	19
3.2	Pohozaev Identities for $(-\Delta)^{1/2}$ in S^1	26
3.3	Pohozaev Identities for the Laplacian in \mathbb{R}^2	32
4	Compactness and Quantization of horizontal 1/2 harmonic maps	33
5	Counter-example	37

1 Introduction

In a recent paper [9] the first and the third authors introduced the notion of *horizontal α harmonic map*, $\alpha \geq 1/2$ with respect to a given C^1 planes distribution. Precisely we consider $P_T \in C^1(\mathbb{R}^m, M_m(\mathbb{R}))$ and $P_N \in C^1(\mathbb{R}^m, M_m(\mathbb{R}))$ such that

$$\left\{ \begin{array}{l} P_T \circ P_T = P_T \quad P_N \circ P_N = P_N \\ P_T + P_N = I_m \\ \forall z \in \mathbb{R}^m \quad \forall U, V \in T_z \mathbb{R}^m \quad \langle P_T U, P_N V \rangle = 0 \\ \|\partial_z P_T\|_{L^\infty(\mathbb{R}^m)} < +\infty \end{array} \right. \quad (2)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^m . In other words P_T is a C^1 map into the orthogonal projections of \mathbb{R}^m . For such a distribution of projections P_T we denote by

$$n := \text{rank}(P_T).$$

Such a distribution identifies naturally with the distribution of n -planes given by the images of P_T (or the Kernel of P_T) and conversely, any C^1 distribution of n -dimensional planes defines uniquely P_T satisfying (2).

For any $\alpha \geq 1/2$ and for $k \geq 1$ we define the space of H^α -Sobolev horizontal maps

$$\mathfrak{H}^\alpha(\mathbb{R}^k) := \{u \in H^\alpha(\mathbb{R}^k, \mathbb{R}^m) \quad ; \quad P_N(u) \nabla u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k)\}$$

Observe that this definition makes sense since we have respectively $P_N \circ u \in H^\alpha(\mathbb{R}^k, M_m(\mathbb{R}))$ and $\nabla u \in H^{\alpha-1}(\mathbb{R}^k, \mathbb{R}^m)$.

Definition 1.1 Given a C^1 plane distribution P_T in \mathbb{R}^m satisfying (2), a map u in the space $\mathfrak{H}^\alpha(\mathbb{R}^k)$ is called **horizontal α -harmonic** with respect to P_T if

$$\forall i = 1 \cdots m \quad \sum_{j=1}^m P_T^{ij}(u)(-\Delta)^\alpha u_j = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k) \quad (3)$$

and we shall use the following notation

$$P_T(u)(-\Delta)^\alpha u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k). \square$$

When the plane distribution P_T is *integrable* that is to say when

$$\forall X, Y \in C^1(\mathbb{R}^m, \mathbb{R}^m) \quad P_N[P_T X, P_T Y] \equiv 0 \quad (4)$$

where $[\cdot, \cdot]$ denotes the Lie Bracket of vector-fields, by using Fröbenius theorem the planes distribution corresponds to the tangent plane distribution of a n -dimensional *foliation* \mathcal{F} , (see e.g [12]). A smooth map u in $\mathfrak{H}^\alpha(\mathbb{R}^m)$ takes values everywhere into a *leaf* of \mathcal{F} that we denote N^n and we are back to the classical theory of α harmonic maps into manifolds. We recall that the notion of weak $1/2$ harmonic maps into a n -dimensional closed manifolds $N^n \subset \mathbb{R}^m$ has been introduced by the first and third author in [7, 8]. These maps are critical points of the fractional energy on \mathbb{R}^k

$$E^{1/2}(u) := \int_{\mathbb{R}^k} |(-\Delta)^{1/4} u|^2 dx^k \quad (5)$$

within

$$H^{1/2}(\mathbb{R}^k, N^n) := \{u \in H^{1/2}(\mathbb{R}^k, \mathbb{R}^m) ; u(x) \in N^n \text{ for a. e. } x \in \mathbb{R}^k\}.$$

The corresponding Euler-Lagrange equation is given by

$$\nu(u) \wedge (-\Delta)^{1/2} u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k), \quad (6)$$

where $\nu(z)$ is the Gauss Maps at $z \in \mathcal{N}$ taking values into the grassmannian $\tilde{Gr}_{m-n}(\mathbb{R}^m)$ of oriented $m - n$ planes in \mathbb{R}^m which is given by the oriented normal $m - n$ -plane to $T_z \mathcal{N}$. One of the main results obtained in [9] is the following

Theorem 1.1 Let P_T be a C^1 distribution of planes (or projections) satisfying (2). Any map $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ (resp. $u \in \mathfrak{H}^1(\mathbb{R}^2)$) satisfying

$$P_T(u)(-\Delta)^{1/2} u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad (7)$$

(resp.

$$P_T(u) \Delta u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2)) \quad (8)$$

is in $\cap_{\delta < 1} C^{0, \delta}(\mathbb{R})$, (resp. $\cap_{\delta < 1} C^{0, \delta}(\mathbb{R}^2)$). \square

In order to prove Theorem 1.1 the authors in [9] use the following two key properties satisfied by respectively horizontal harmonic and horizontal 1/2-harmonic maps.

Horizontal harmonic maps satisfy an elliptic Schrödinger type system with an anti-symmetric potential $\Omega \in L^2(\mathbb{R}^k, \mathbb{R}^k \otimes so(m))$ of the form

$$-\Delta u = \Omega(P_T) \cdot \nabla u. \quad (9)$$

Hence, following the analysis in [15] the authors deduced in two dimension the local existence on a disc D^2 of $A(P_T) \in L^\infty \cap W^{1,2}(D^2, Gl_m(\mathbb{R}))$ and $B(P_T) \in W^{1,2}(D^2, M_m(\mathbb{R}))$ such that

$$\operatorname{div}(A(P_T) \nabla u) = \nabla^\perp B(P_T) \cdot \nabla u \quad (10)$$

from which the regularity of u can be deduced using Wente's *Integrability by compensation* which can be summarized in the following estimate

$$\|\nabla^\perp B \cdot \nabla u\|_{H^{-1}(D^2)} \leq C \|\nabla B\|_{L^2(D^2)} \|\nabla u\|_{L^2(D^2)}. \quad (11)$$

A similar property is satisfied by horizontal 1/2-harmonic maps. Precisely in [9] conservation laws corresponding to (10) but for general *horizontal 1/2-harmonic maps* have been discovered: locally, modulo some smoother terms coming from the application of non-local operators on cut-off functions, the authors construct $A(P_T) \in L^\infty \cap H^{1/2}(\mathbb{R}, Gl_m(\mathbb{R}))$ and $B(P_T) \in H^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$ such that

$$(-\Delta)^{1/4}(A(P_T)v) = \mathcal{J}(B(P_T), v) + \text{cut-off}, \quad (12)$$

where $v := (P_T(-\Delta)^{1/4}v, \mathcal{R}(P_N(-\Delta)^{1/4}v))$ and \mathcal{R} denotes the Riesz operator and \mathcal{J} is a bilinear pseudo-differential operator satisfying

$$\|\mathcal{J}(B, v)\|_{H^{-1/2}(\mathbb{R})} \leq C \|(-\Delta)^{1/4}B\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}. \quad (13)$$

Moreover by assuming that $P_T \in C^2(\mathbb{R}^m)$ and $\|\partial_z P_T\|_{L^\infty(\mathbb{R})} < +\infty$ and by bootstrapping arguments one gets that every horizontal 1/2 harmonic map $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ is $C_{loc}^{1,\alpha}(\mathbb{R})$, for every $\alpha < 1$, (see [5]).

We also remark that if $\Pi: S^1 \setminus \{-i\} \rightarrow \mathbb{R}$, $\Pi(\cos(\theta) + i \sin(\theta)) = \frac{\cos(\theta)}{1 + \sin(\theta)}$ is the classical stereographic projection whose inverse is given by

$$\Pi^{-1}(x) = \frac{2x}{1+x^2} + i \left(-1 + \frac{2}{1+x^2} \right). \quad (14)$$

then the following relation between the 1/2 Laplacian in \mathbb{R} and in S^1 holds:

Proposition 1.1 (Proposition 4.1, [6]) *Given $u: \mathbb{R} \rightarrow \mathbb{R}^m$ set $v := u \circ \Pi: S^1 \rightarrow \mathbb{R}^m$. Then $u \in L_{\frac{1}{2}}(\mathbb{R})^{(1)}$ if and only if $v \in L^1(S^1)$. In this case*

$$(-\Delta)_{S^1}^{\frac{1}{2}} v(e^{i\theta}) = \frac{((-\Delta)_{\mathbb{R}}^{\frac{1}{2}} u)(\Pi(e^{i\theta}))}{1 + \sin \theta} \quad \text{in } \mathcal{D}'(S^1 \setminus \{-i\}), \quad (15)$$

⁽¹⁾We recall that $L_{\frac{1}{2}}(\mathbb{R}) := \left\{ u \in L_{loc}^1(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1+x^2} dx < \infty \right\}$

Observe that $1 + \sin(\theta) = |\Pi'(\theta)|$, and hence we have

$$\int_{S^1} (-\Delta)^{\frac{1}{2}} v(e^{i\theta}) \varphi(e^{i\theta}) d\theta = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{2}} u(x) \varphi \circ \Pi^{-1}(x) dx \quad \text{for every } \varphi \in C_0^\infty(S^1 \setminus \{-i\}).$$

From Proposition 1.1 it follows that $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ is a horizontal 1/2 harmonic map in \mathbb{R} if and only if $v := u \circ \Pi \in \mathfrak{H}^{1/2}(S^1)$ is a horizontal 1/2 harmonic map in S^1 .

The goal of this paper is to investigate compactness and quantization properties of sequences of horizontal 1/2 harmonic maps $u_k \in \mathfrak{H}^{1/2}(\mathbb{R})$. Our main result is the following:

Theorem 1.2 *Let $u_k \in \mathfrak{H}^{1/2}(\mathbb{R})$ be a sequence of horizontal 1/2-harmonic maps such that*

$$\|u_k\|_{\dot{H}^{1/2}} \leq C, \quad \|(-\Delta)^{1/2} u_k\|_{L^1} \leq C. \quad (16)$$

Then it holds:

1. *There exist $u_\infty \in \mathfrak{H}^{1/2}(\mathbb{R})$ and a possibly empty set $\{a_1, \dots, a_\ell\}$, $\ell \geq 1$, such that up to subsequence*

$$u_k \rightarrow u_\infty \quad \text{in } \dot{W}_{loc}^{1/2,p}(\mathbb{R} \setminus \{a_1, \dots, a_\ell\}), \quad p \geq 2 \text{ as } k \rightarrow +\infty \quad (17)$$

and

$$P_T(u_\infty)(-\Delta)^{1/2} u_\infty = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (18)$$

2. *There is a family $\tilde{u}_\infty^{i,j} \in \mathfrak{H}^{1/2}(\mathbb{R})$ of horizontal 1/2-harmonic maps ($i \in \{1, \dots, \ell\}, j \in \{1, \dots, N_i\}$), such that up to subsequence*

$$\left\| (-\Delta)^{1/4} \left(u_k - u_\infty - \sum_{i,j} \tilde{u}_\infty^{i,j}((x - x_{i,j}^k)/r_{i,j}^k) \right) \right\|_{L_{loc}^2(\mathbb{R})} \rightarrow 0, \quad \text{as } k \rightarrow +\infty. \quad (19)$$

for some sequences $r_{i,j}^k \rightarrow 0$ and $x_{i,j}^k \in \mathbb{R}$.

We would like to make some comments and remarks on Theorem 1.2.

1. We first mention that the condition $\|(-\Delta)^{1/2} u_k\|_{L^1} \leq C$ is always satisfied in the case the maps u_k take values into a closed manifold of \mathbb{R}^m (case of sequences of 1/2 harmonic maps) as soon as $\|u_k\|_{\dot{H}^{1/2}} \leq C$. This follows from the fact that if u is a 1/2-harmonic maps with values into a closed manifold of \mathcal{N}^n of \mathbb{R}^m then the following inequality holds (see Proposition 4.1)

$$\|(-\Delta)^{1/2} u\|_{L^1(\mathbb{R})} \leq C \|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})}^2. \quad (20)$$

Hence we have the following corollary

Corollary 1.1 *Let \mathcal{N}^n be a closed C^2 submanifold of \mathbb{R}^m and let $u_k \in H^{1/2}(\mathbb{R}, \mathcal{N}^n)$ be a sequence of $1/2$ -harmonic maps such that*

$$\|u_k\|_{\dot{H}^{1/2}} \leq C \quad (21)$$

then the conclusions of theorem 1.2 hold. In particular modulo extraction of a subsequence we have the following energy identity:

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}} |(-\Delta)^{1/4} u_k|^2 dx = \int_{\mathbb{R}} |(-\Delta)^{1/4} u_\infty|^2 dx + \sum_{i,j} \int_{\mathbb{R}} |(-\Delta)^{1/4} \tilde{u}_\infty^{i,j}|^2 dx \quad (22)$$

where $\tilde{u}_\infty^{i,j}$ are the **bubbles** associated to the weak convergence.

For the moment it remains open to know whether the bound (20) holds or not in the general case of horizontal $1/2$ -harmonic maps.

2. The compactness issue (first part of Theorem 1.2) is quite standard. The most delicate part is the quantization analysis consisting in verifying that there is no dissipation of the energy in the region between u_∞ and the *bubbles* $\tilde{u}_\infty^{i,j}$ and between the bubbles themselves (the so-called *neck-regions*). The strategy of the proof of theorem 1.2 is presented in the next section. One important tool we are using for proving theorem 1.2 is a new **Pohozaev identity** for the half Laplacian in 1 dimension.

Theorem 1.3 [Pohozaev Identity in \mathbb{R}] *Let $u \in W^{1,2}(\mathbb{R}, \mathbb{R}^m)$ be such that*

$$\frac{du}{dx} \cdot (-\Delta)^{1/2} u = 0 \quad \text{a.e in } \mathbb{R}. \quad (23)$$

Assume that

$$\int_{\mathbb{R}} |u - u_0| dx < +\infty, \quad \int_{\mathbb{R}} \left| \frac{du}{dx}(x) \right| dx < +\infty \quad (24)$$

Then the following identity holds

$$\left| \int_{x \in \mathbb{R}} \frac{x^2 - t^2}{(x^2 + t^2)^2} u(x) dx \right|^2 = \left| \int_{x \in \mathbb{R}} \frac{2xt}{(x^2 + t^2)^2} u(x) dx \right|^2. \quad (25)$$

We observe that the conditions (24) are satisfied by the $1/2$ -harmonic maps with valued into a closed C^2 sub-manifold. By means of the stereographic projection we get an analogous formula in S^1 .

Theorem 1.4 [Pohozaev Identity on S^1] *Let u be a $W^{1,2}$ map from S^1 into \mathbb{R}^m satisfying*

$$\frac{du}{d\theta} \cdot (-\Delta)^{1/2} u = 0 \quad \text{a. e. on } S^1 \quad (26)$$

then the following identity holds

$$\left| \int_0^{2\pi} u(\theta) \cos \theta \, d\theta \right|^2 = \left| \int_0^{2\pi} u(\theta) \sin \theta \, d\theta \right|^2 \quad (27)$$

□

We have now to give some explanations why these identities belong to the *Pohozaev identities* family. These identities are produced by the conformal invariance of the highest order derivative term in the Lagrangian from which the Euler Lagrange is issued : in 2D for instance

$$E(u) = \int_{\mathbb{R}^2} |\nabla u|^2 dx^2$$

is conformal invariant, whereas

$$E^{1/2}(u) = \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 \, dx$$

is conformal invariant in 1D. The infinitesimal perturbations issued from the dilations produce the following infinitesimal variations of these highest order terms respectively

$$\sum_{i=1}^2 x_i \frac{\partial u}{\partial x_i} \cdot \Delta u \quad \text{in 2D} \quad \text{and} \quad x \frac{du}{dx} \cdot (-\Delta)^{1/2} u \quad \text{in 1D}$$

Being a critical point respectively of E in 2D and $E^{1/2}$ in 1D and assuming enough regularity gives respectively

$$\sum_{i=1}^2 x_i \frac{\partial u}{\partial x_i} \cdot \Delta u = 0 \quad \text{in 2D} \quad \text{and} \quad x \frac{du}{dx} \cdot (-\Delta)^{1/2} u = 0 \quad \text{in 1D}$$

In two dimensions, integrating this identity on a ball $B(x_0, r)$ gives the following **balancing law** between the **radial** part and the **angular** part of the energy classically known as **Pohozaev identity**.

Theorem 1.5 *Let $u \in W^{2,2}(\mathbb{R}^2, \mathbb{R}^m)$ such that*

$$\sum_{i=1}^2 x_i \frac{\partial u}{\partial x_i} \cdot \Delta u = 0 \quad \text{a.e in } B(0, 1). \quad (28)$$

Then it holds

$$\int_{\partial B(x_0, r)} \left| \frac{1}{r} \frac{\partial u}{\partial \theta} \right|^2 d\theta = \int_{\partial B(x_0, r)} \left| \frac{\partial u}{\partial r} \right|^2 d\theta \quad (29)$$

for all $r \in [0, 1]$.

In 1 dimension one might wonder what corresponds to the 2 dimensional dichotomy between **radial** and **angular** parts. We illustrate below the correspondence of dichotomies respectively in 1 and 2 dimensions.

$$\begin{array}{ccc}
2\text{D} & \longleftrightarrow & 1\text{D} \\
\text{radial : } \frac{\partial u}{\partial r} & \longleftrightarrow & \text{symmetric part of } u \quad : \quad u^+(x) := \frac{u(x)+u(-x)}{2} \\
\text{angular : } \frac{\partial u}{\partial \theta} & \longleftrightarrow & \text{antisymmetric part of } u \quad : \quad u^-(x) := \frac{u(x)-u(-x)}{2}
\end{array}$$

Observe moreover that our **Pohozaev identity in 1D** (27) can be rewritten as a **balancing law** between the **symmetric** part and the **antisymmetric** part of u .

$$\left| \int_0^{2\pi} u^+(\theta) \cos \theta \, d\theta \right|^2 = \left| \int_0^{2\pi} u^-(\theta) \sin \theta \, d\theta \right|^2 \quad (30)$$

This law is not invariant under the action of the Möbius group but the condition (26) is. Applying for instance rotations by an arbitrary angle $\alpha \in \mathbb{R}$, the identity (27) implies

$$\begin{cases} |u_1| = |u_{-1}| \\ u_1 \cdot u_{-1} = 0 \end{cases} \quad (31)$$

where

$$\begin{cases} u_1 := \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \cos \theta \, d\theta \\ u_{-1} = \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \sin \theta \, d\theta \end{cases}$$

The previous implies that there are “as many” Pohozaev identities as elements in this group minus the action of rotations, that is there are a $3-1=2=D+1$ dimensional family of identities exactly as in the 2-D case where there are exactly as many **Pohozaev identities** (29) as choices of center $x_0 \in \mathbb{R}^2$ and radius $r > 0$ (which is again a $D+1 = 3$ dimensional space).

1.1 The strategy of the proof of theorem 1.2.

We first recall the definitions of a bubble and a neck region.

Definition 1.2 (Bubble) *A Bubble is a non-constant horizontal $1/2$ -harmonic map $u \in \mathfrak{H}^{1/2}(\mathbb{R})$.*

Definition 1.3 (Neck region) A neck region for a sequence $f_k \in L^2(\mathbb{R})$ is the union of finite degenerate annuli of the type $A_k(x) = B(x, R_k) \setminus B(x, r_k)$ with $r_k \rightarrow 0$ and $\frac{R_k}{r_k} \rightarrow +\infty$ as $k \rightarrow +\infty$ such that

$$\lim_{\Lambda \rightarrow \infty} \lim_{k \rightarrow \infty} \left(\sup_{\rho \in [\Lambda r_k, (2\Lambda)^{-1} R_k]} \int_{B(x, 2\rho) \setminus B(x, \rho)} |f_k|^2 dx \right)^{1/2} = 0. \quad (32)$$

The main achievement of the present work is to show, under the assumptions of theorem 1.2, that (32) for $f_k := (-\Delta)^{1/4} u_k$ can be improved to

$$\lim_{\Lambda \rightarrow \infty} \lim_{k \rightarrow \infty} \left(\int_{B(x, \frac{R_k}{\Lambda}) \setminus B(x, \Lambda r_k)} |f_k|^2 dx \right)^{1/2} = 0. \quad (33)$$

The proof of estimate of (33) will be the aim of Section 4.

3. Theorem 1.2 has been proved by the first author in [4] in the case of sequences of 1/2-harmonic maps u_k with values into the \mathcal{S}^{m-1} sphere, see also [13]. In this case in order to prove (33) we use the duality of the Lorentz spaces $L^{2,1} - L^{2,\infty}$.⁽²⁾

We first show that the $L^{2,\infty}$ norm of the u_k is arbitrary small in the *neck region* and then we use the fact that $L^{2,1}$ norm of 1/2-harmonic maps with values into a sphere is uniformly globally bounded. This last global estimate follows directly by the formulation of the 1/2-harmonic maps equation that the first and third author discovered in [7] in terms of special algebraic quantities (three-commutators) satisfying particular integrability compensation properties.

The fact that the $L^{2,\infty}$ norm of a sequence of horizontal 1/2-harmonic maps u_k is arbitrary small in *neck regions* still holds in the general case, precisely we have:

Lemma 1.1 ($L^{2,\infty}$ estimates) *There exists $\delta > 0$ such that for any sequence of horizontal 1/2-harmonic maps $u_k \in \mathfrak{H}^{1/2}(\mathbb{R})$ satisfying*

$$\sup_{\rho \in [\Lambda r_k, (2\Lambda)^{-1} R_k]} \left(\int_{B(0, 2\rho) \setminus B(0, \rho)} |(-\Delta)^{1/4} u_k|^2 dx \right)^{1/2} < \delta$$

with $r_k \rightarrow 0$ and $\frac{R_k}{r_k} \rightarrow +\infty$ as $k \rightarrow +\infty$ and for all $\Lambda > 1$ with $\Lambda r_k < (2\Lambda)^{-1} R_k$ then

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow \infty} \|(-\Delta)^{1/4} u\|_{L^{2,\infty}(B(0, (2\Lambda)^{-1} R_k) \setminus B(0, \Lambda r_k))} = 0. \quad (34)$$

Unfortunately we do not know if such a global uniform $L^{2,1}$ estimate exists even in the case of 1/2-harmonic maps with values into a closed submanifold of \mathbb{R}^m .

To overcome of this lack of a global $L^{2,1}$ estimate, we show a precise asymptotic development of $(-\Delta)^{1/4} u_k$ in annuli $A_{r_k, R_k} = B(x, R_k) \setminus B(x, r_k)$ where the L^2 norm of $(-\Delta)^{1/4} u_k$ is small. To get such an asymptotic development we argue as follows. We first prove the following result (for simplicity we will consider an annulus centered at $x = 0$).

⁽²⁾ see section 1.2 for a definition.

Theorem 1.6 *Let $u_k \in \mathfrak{H}^{1/2}(\mathbb{R})$ be a sequence of horizontal $1/2$ -harmonic maps satisfying the assumptions of Theorem 1.1. There exists $\delta > 0$ such that if*

$$\|(-\Delta)^{1/4}u_k\|_{L^2(B(0,R_k)\setminus B(0,r_k))} < \delta, \quad (35)$$

with $r_k \rightarrow 0$ and $\frac{R_k}{r_k} \rightarrow +\infty$ as $k \rightarrow +\infty$, then for all $\Lambda > 1$ with $\Lambda r_k < (2\Lambda)^{-1}R_k$ and $x \in B(0, (2\Lambda)^{-1}R_k) \setminus B(0, \Lambda r_k)$ we have

$$(-\Delta)^{1/4}u_k(x) = \frac{a_k^+(x)\overrightarrow{c_{r_k}}}{|x|^{1/2}} + h_k + g_k, \quad (36)$$

where $g_k \in L^2(\mathbb{R})$ with $\limsup_{k \rightarrow \infty} \|g_k\|_{L^2} = 0$,

$$\overrightarrow{c_{r_k}} = O\left(\left(\log\left(\frac{R_k}{2\Lambda^2 r_k}\right)\right)^{-1/2}\right), \quad \text{as } k \rightarrow +\infty, \Lambda \rightarrow +\infty \quad (37)$$

$a_k^+ \in L^\infty \cap \dot{H}^{1/2}(\mathbb{R}, Gl_m(\mathbb{R}))$, $h_k \in L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})$,

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|h_k\|_{L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})} < \infty \quad (38)$$

and

$$\|a_k\|_{\dot{H}^{1/2}} + \|a_k\|_{L^\infty} \leq C \|(-\Delta)^{1/4}u_k\|_{L^2(\mathbb{R})}. \quad (39)$$

From Theorem 1.6 we will deduce that in a neck region (centered for simplicity of notation in $x = 0$) we have

$$\lim_{\Lambda \rightarrow \infty} \lim_{k \rightarrow \infty} \left(\int_{B(0, \frac{R_k}{2\Lambda_k}) \setminus B(0, \Lambda_k r_k)} |((-\Delta)^{1/4}u_k)^-|^2 dx \right)^{1/2} = 0. \quad (40)$$

where for a given function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ we denote by f^+ and f^- respectively the symmetric and antisymmetric part of f with respect to the origin, i.e. $f^+(x) = \frac{f(x)+f(-x)}{2}$ and $f^-(x) = \frac{f(x)-f(-x)}{2}$. It remains to find a link between the symmetric and the antisymmetric part of $(-\Delta)^{1/4}u_k$. To this purpose we make use of the **Pohozaev type formula** theorem 1.3, which is one of the main result of this paper.

We explain below in some steps how formula (25) permits to get an information of the L^2 norm of $((-\Delta)^{1/4}u_k)^-$.

- We observe that we can rewrite the l.h.s and r.h.s of (25) respectively as follows

$$\left| \int_{x \in \mathbb{R}} \frac{x^2 - t^2}{(x^2 + t^2)^2} u^+(x) dx \right|^2 = t^{-2} \left| \int_{x \in \mathbb{R}} (-\Delta)^{-1/4} \left[\frac{x^2 - 1}{(x^2 + 1)^2} \right] ((-\Delta)^{1/4}u)^+(xt) dx \right|^2.$$

$$\left| \int_{x \in \mathbb{R}} \frac{2xt}{(x^2 + t^2)^2} u(x) dx \right|^2 = t^{-2} \left| \int_{x \in \mathbb{R}} (-\Delta)^{-1/4} \left[\frac{2x}{(x^2 + 1)^2} \right] ((-\Delta)^{1/4} u)^-(xt) dx \right|^2.$$

Therefore we can rewrite the formula (25) as

$$\begin{aligned} & \left| \int_{x \in \mathbb{R}} (-\Delta)^{-1/4} \left[\frac{x^2 - 1}{(x^2 + 1)^2} \right] ((-\Delta)^{1/4} u)^+(xt) \right|^2 \\ &= \left| \int_{x \in \mathbb{R}} (-\Delta)^{-1/4} \left[\frac{2x}{(x^2 + 1)^2} \right] ((-\Delta)^{1/4} u)^-(xt) dx \right|^2. \end{aligned} \quad (41)$$

- We prove that

$$M^+[v](t) := \int_{x \in \mathbb{R}} (-\Delta)^{-1/4} \left[\frac{x^2 - 1}{(x^2 + 1)^2} \right] v(xt) dx$$

and

$$M^-[v](t) := \int_{x \in \mathbb{R}} (-\Delta)^{-1/4} \left[\frac{2x}{(x^2 + 1)^2} \right] v(xt) dx$$

are isomorphisms from $L_+^p(\mathbb{R})$ onto $L_+^p(\mathbb{R})$ (resp. $L_-^p(\mathbb{R})$ onto $L_-^p(\mathbb{R})$) for every $p > 1$ and by interpolation from $L_+^{2,1}(\mathbb{R})$ onto $L_+^{2,1}(\mathbb{R})$ (resp. $L_-^{2,1}(\mathbb{R})$ onto $L_-^{2,1}(\mathbb{R})$). Here L_+^p is the space of even L^p functions and L_-^p the space of odd L^p functions.

- By plugging the development (36) into (41) we obtain an information also of the asymptotic behaviour of the L^2 norm of $((-\Delta)^{1/4} u_k)^+$ in an annular region, namely for all $\Lambda > 1$ with $\Lambda r_k < (2\Lambda)^{-1} R_k$ and $x \in B(0, (2\Lambda)^{-1} R_k) \setminus B(0, \Lambda r_k)$

$$\int_{B(0, \frac{R_k}{2\Lambda_k}) \setminus B(0, \Lambda_k r_r)} |((-\Delta)^{1/4} u_k)^-|^2 dx = \tilde{h}_k + \tilde{g}_k, \quad (42)$$

where $\tilde{g}_k \in L^2(\mathbb{R})$ with $\limsup_{k \rightarrow \infty} \|\tilde{g}_k\|_{L^2} = 0$ and

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|\tilde{h}_k\|_{L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1} R_k})} < \infty.$$

- By combining (36) and (42) we get

Theorem 1.7 *Under the assumptions of Theorem 1.6, we have*

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|(-\Delta)^{1/4} u_k\|_{L^2(B(0, (2\Lambda)^{-1} R_k) \setminus B(0, \Lambda_k r_r))} = 0. \quad (43)$$

We would like to establish a link between the above results and the result obtained by the second and third author [14] in the framework of harmonic maps in $2D$ with values into a closed manifold and that can easily be extended to horizontal harmonic maps. Also in this case, the strategy has been to prove an $L^{2,1}$ -estimate on the angular part of the

gradient (which play the role of the antisymmetric part $(-\Delta)^{1/4}u$). Precisely suppose we have a sequence $u_k \in \mathfrak{H}^1(D^2)$ of horizontal harmonic maps such that $\|\nabla u_k\|_{L^2(D^2)} \leq C$. Then they satisfy

$$-\Delta u_k = \Omega_k(P_T)\nabla u \text{ in } \mathcal{D}'(D^2), \quad (44)$$

where $\Omega \in L^2(D^2, \mathbb{R}^2 \otimes so(m))$ is an antisymmetric potential. In [14] the authors found the following asymptotic development for ∇u_k in an annular domain $B(0, R_k) \setminus B(0, r_k)$ where $\|\Omega_k\|_{L^2(B(0, R_k) \setminus B(0, r_k))}$ is *small*:

$$\nabla u_k(x) = \frac{a_k(|x|)\vec{c}_{r_k}}{|x|^{1/2}} + h_k, \quad (45)$$

where a_k is radial, $a_k \in L^\infty(B(0, R_k) \setminus B(0, r_k))$, $h_k \in L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})$ (for every $\Lambda > 1$ such that $\Lambda r_k < (2\Lambda)^{-1}R_k$),

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|h_k\|_{L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})} < \infty, \quad \|a_k\|_{L^\infty} \leq C\|\nabla u_k\|_{L^2}. \quad (46)$$

From (45) they deduce that

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \left\| \frac{1}{\rho} \frac{\partial u_k}{\partial \theta} \right\|_{L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})} < \infty. \quad (47)$$

They also prove that the $L^{2,\infty}$ of ∇u_k is arbitrary small in degenerating annuli, namely

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|\nabla u_k\|_{L^{2,\infty}(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})} = 0. \quad (48)$$

By combining (47) and (48) they obtain

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \left\| \frac{1}{\rho} \frac{\partial u_k}{\partial \theta} \right\|_{L^2(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})} = 0. \quad (49)$$

The Pohozaev identity in $2D$ (29) obtained from (28) implies that there is no loss of energy in the neck region:

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|\nabla u_k\|_{L^2(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})} = 0. \quad (50)$$

In a last part, we give a counter example to quantization for general sequence of Schrödinger equation with antisymmetric potential. For the Laplacian it has been done in [14]. Hence here we give an example of sequence of u_k satisfying

$$(-\Delta)^{\frac{1}{2}}u_k = \Omega_k u_k + \Omega_{1k} u_k$$

with Ω_k antisymmetric whose L^2 -norm goes to zero in a neck region, and the $L^{2,1}$ -norm Ω_{1k} goes to zero in the neck region, despite the L^2 -norm of u_k remains bounded from below. As in the case of the Laplacian in [14], we take a modification of the Green function. Since our functions will be even, this insures that that no Pohozaev identity can occur.

1.2 Definitions and Notations

In this section we introduce some definitions and notations that will be used in the paper.

1. Given $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ we denote by $\hat{\varphi}$ or $\mathcal{F}[\varphi]$ the Fourier transform of φ and by $\check{\varphi}$ or $\mathcal{F}^{-1}[\varphi]$ the inverse Fourier transform of φ .
2. Given $u, v \in \mathbb{R}^n$ we denote by $\langle u, v \rangle$ the standard scalar product of u and v .
3. Given $z \in \mathbb{C}$ we denote by $\Re z$ and $\Im z$ the real and imaginary part of z .

Next we introduce some functional spaces.

First, we will note L_+^p the space of even L^p functions and L_-^p the space of odd L^p functions.

We denote by $L^{2,\infty}(\mathbb{R}^n)$ the space of measurable functions f such that

$$\sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| \geq \lambda\}|^{1/2} < +\infty,$$

and $L^{2,1}(\mathbb{R}^n)$ is the space of measurable functions satisfying

$$\int_0^{+\infty} |\{x \in \mathbb{R}^n : |f(x)| \geq \lambda\}|^{1/2} d\lambda < +\infty.$$

We can check that $L^{2,\infty}$ and $L^{2,1}$ forms a duality pair.

We denote by $\mathcal{H}^1(\mathbb{R}^n)$ the Hardy space which is the space of L^1 functions f on \mathbb{R}^n satisfying

$$\int_{\mathbb{R}^n} \sup_{t>0} |\phi_t * f|(x) dx < +\infty,$$

where $\phi_t(x) := t^{-n} \phi(t^{-1}x)$ and ϕ is some function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \phi(x) dx = 1$.

Finally for every $s \in \mathbb{R}$ and $q > 1$ we denote by $\dot{W}^{s,q}(\mathbb{R}^n)$ the fractional Sobolev space

$$\{f \in \mathcal{S}' : \mathcal{F}^{-1}[|\xi|^s |\mathcal{F}[f]|] \in L^q(\mathbb{R}^n)\}.$$

For more properties on the Lorentz spaces, Hardy space \mathcal{H}^1 and fractional Sobolev spaces we refer to [10] and [11].

The paper is organized as follows. In Section 2 we analyze the asymptotic behaviour of the solutions of special nonlocal Schrödinger systems with a L^2 antisymmetric potential in degenerate annular domain. In Section 3 we describe two new Pohozaev type formulae,

one on \mathbb{R} and one in S^1 , which can be obtained one from the other by means of the stereographic projections and which are satisfied in particular by 1/2-harmonic maps respectively on \mathbb{R} and on S^1 . We also compare such formulae with a Pohozev formula in $2D$ which is satisfied by harmonic maps. In Section 4 we deduce from the results of Section 2 and 3 the quantization of horizontal 1/2-harmonic maps. Finally in Section 5 we describe an example showing that in general we cannot have quantization for solutions of a general nonlocal Schrödinger systems with a L^2 antisymmetric potential.

2 Preliminary results on nonlocal Schrödinger system

As we will see later horizontal 1/2-harmonic maps in 1-D satisfy special nonlocal system of the form

$$(-\Delta)^{1/4}v = \Omega v + \Omega_1 v + \mathcal{Z}(Q, v) + g \quad (51)$$

where $v \in L^2(\mathbb{R})$, $Q \in H^{1/2}(\mathbb{R})$, $\Omega \in L^2(\mathbb{R}, so(m))$, $\Omega_1 \in L^{2,1}(\mathbb{R})$, $g \in L^1(\mathbb{R})$ and $\mathcal{Z}: H^{1/2}(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$ satisfies the following stability property: if $Q, P \in \dot{H}^{1/2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $v \in L^2$ then

$$P\mathcal{Z}(Q, v) = \mathcal{A}_{\mathcal{Z}}(P, Q)v + J_{\mathcal{Z}}(P, Q, v), \quad (52)$$

where

$$\|\mathcal{A}_{\mathcal{Z}}(P, Q)\|_{L^{2,1}(\mathbb{R})} \leq C \|(-\Delta)^{1/4}[P]\|_{L^2(\mathbb{R})} \|(-\Delta)^{1/4}[Q]\|_{L^2(\mathbb{R})},$$

and

$$\|J_{\mathcal{Z}}(P, Q, v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C \left(\|(-\Delta)^{1/4}[P]\|_{L^2(\mathbb{R})} + \|(-\Delta)^{1/4}[Q]\|_{L^2(\mathbb{R})} \right) \|v\|_{L^2(\mathbb{R})}.$$

Actually \mathcal{Z} is a linear combination of the following *pseudo-differential operators*:

$$T(Q, v) := (-\Delta)^{1/4}(Qv) - Q(-\Delta)^{1/4}v + (-\Delta)^{1/4}Qv \quad (53)$$

and

$$S(Q, v) := (-\Delta)^{1/4}[Qv] - \mathcal{R}(Q\mathcal{R}(-\Delta)^{1/4}v) + \mathcal{R}((-\Delta)^{1/4}Q\mathcal{R}v) \quad (54)$$

$$F(Q, v) := \mathcal{R}[Q]\mathcal{R}[v] - Qv. \quad (55)$$

$$\Lambda(Q, v) := Qv + \mathcal{R}[Q\mathcal{R}[v]]. \quad (56)$$

In [4] the first author improve the estimates on the operators T, S obtained in [7]:

Theorem 2.1 *Let $v \in L^2(\mathbb{R})$, $Q \in \dot{H}^{1/2}(\mathbb{R})$. Then $T(Q, v), S(Q, v) \in \mathcal{H}^1(\mathbb{R})$ and*

$$\|T(Q, v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}. \quad (57)$$

$$\|S(Q, v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C \|Q\|_{\dot{H}^{1/2}(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}. \quad \square \quad (58)$$

As a consequence of the Coifman-Rochberg-Weiss estimate [2] we also have

$$\|F(f, v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}. \quad (59)$$

for $f, v \in L^2$.

In [9] the first and the third author show that the stability property (52) holds for the operators (53), (54), (55).

Moreover the authors have shown that if the L^2 norm of Ω is “small” then the system (51) is equivalent to a conservation law:

Theorem 2.2 (Theorem 3.11 in [9]) *Let $v \in L^2(\mathbb{R}, \mathbb{R}^m)$ be a solution of (51), where $\Omega \in L^2(\mathbb{R}, so(m))$, $\Omega_1 \in L^{2,1}(\mathbb{R})$, $\mathcal{Z}(Q, v) \in \mathcal{H}^1$ for every $Q \in H^{1/2}$, $v \in L^2$ with*

$$\|\mathcal{Z}(Q, v)\|_{\mathcal{H}^1} \leq C\|Q\|_{H^{1/2}}\|v\|_{L^2},$$

and $\mathcal{Z}(Q, v)$ satisfies (52). There exists $\gamma > 0$ such that if $\|\Omega\|_{L^2} < \gamma$, then there exist $A = A(\Omega, \Omega_1, Q) \in L^\infty \cap H^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$ and an operator $B = B(\Omega, \Omega_1, Q) \in H^{1/2}(\mathbb{R})$ such that

$$\|A\|_{H^{1/2}} + \|B\|_{H^{1/2}} \leq C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{H^{1/2}}) \quad (60)$$

$$dist(\{A, A^{-1}\}, SO(m)) \leq C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{H^{1/2}}) \quad (61)$$

and

$$(-\Delta)^{1/4}[Av] = \mathcal{J}(B, v) + Ag, \quad (62)$$

where \mathcal{J} is a linear operator in B, v , $\mathcal{J}(B, v) \in \mathcal{H}^1(\mathbb{R})$ and

$$\|\mathcal{J}(B, v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C\|B\|_{H^{1/2}}\|v\|_{L^2}. \quad (63)$$

In the following result we show that if the L^2 -norm of the antisymmetric potential Ω in (51) is *small* in an annular domain $A_{r,R} := B(0, R) \setminus B(0, r)$ then a precise asymptotic development inside $A_{r,R}$ holds for the solution v to (51).

We assume for simplicity that $g \equiv 0$.

Proposition 2.1 *Let $v \in L^2(\mathbb{R}, \mathbb{R}^m)$ be a solution of*

$$(-\Delta)^{1/4}v = \Omega v + \Omega_1 v + \mathcal{Z}(Q, v) \quad (64)$$

where $\Omega \in L^2(\mathbb{R}, so(m))$, with $\|\Omega\|_{L^2(B(0,R) \setminus B(0,r))} < \gamma$, $\Omega_1 \in L^{2,1}(\mathbb{R})$, $\mathcal{Z}(Q, v) \in \mathcal{H}^1$ for every $Q \in H^{1/2}$, $v \in L^2$ with

$$\|\mathcal{Z}(Q, v)\|_{\mathcal{H}^1} \leq C\|Q\|_{H^{1/2}}\|v\|_{L^2},$$

and $\mathcal{Z}(Q, v)$ satisfies (52) Then there exists $A = A(\Omega, \Omega_1, Q) \in H^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$ with

$$\begin{aligned} \|A\|_{H^{1/2}} + \|A^{-1}\|_{H^{1/2}} &\leq C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{H^{1/2}}) \\ dist(\{A, A^{-1}\}, SO(m)) &\leq C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{H^{1/2}}), \end{aligned} \quad (65)$$

and $\vec{c}_r \in \mathbb{R}^m$ such that for every $x \in B(0, R) \setminus B(0, r)$ and for every $\Lambda > 2$ with $\Lambda r < (2\Lambda)^{-1}R$ it holds

$$v(x) = (A^{-1}(x))^+ \vec{c}_r \frac{1}{|x|^{1/2}} + h(x) + g(x) \quad (66)$$

with $h \in L^{2,1}(A_{\Lambda r, (2\Lambda)^{-1}R})$, $\|h\|_{L^{2,1}(A_{\Lambda r, (2\Lambda)^{-1}R})} \leq C\|v\|_{L^2}$ where C is a positive constant independent of r, R, Λ and $g \in L^2(\mathbb{R})$ with $\|g\|_{L^2(\mathbb{R})} \leq C|\vec{c}_r|$.

Proof of Proposition 2.1. We split Ω as follows:

$$\Omega = \Omega^r + \Omega^R + \Omega^{R,r}$$

where

$$\begin{aligned} \Omega^{R,r} &= \mathbb{1}_{B(0,R) \setminus B(0,r)} \Omega \\ \Omega^r &= \mathbb{1}_{B(0,r)} \Omega \\ \Omega^R &= \mathbb{1}_{B^c(0,R)} \Omega. \end{aligned}$$

We write the system (64) in the following form

$$(-\Delta)^{1/4} v = \Omega^{R,r} v + \Omega_1 v + \mathcal{Z}(Q, v) + h_r + h_R \quad (67)$$

where

$$h_r = \Omega^r v, \quad \text{and} \quad h_R = \Omega^R v.$$

Observe that $h_r, h_R \in L^1(\mathbb{R})$ with $\text{supp}(h_R) \subseteq B^c(0, R)$, $\text{supp}(h_r) \subseteq B(0, r)$, and $\|h_r\|_{L^1} + \|h_R\|_{L^1} \leq C(\|v\|_{L^2})$. From Theorem 2.2 there exist $A = A(\Omega, \Omega_1, Q) \in \dot{H}^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$ and an operator $B = B(\Omega, \Omega_1, Q) \in \dot{H}^{1/2}(\mathbb{R})$ such that

$$\|A\|_{\dot{H}^{1/2}} + \|A^{-1}\|_{\dot{H}^{1/2}} + \|B\|_{\dot{H}^{1/2}} \leq C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{\dot{H}^{1/2}}) \quad (68)$$

$$\text{dist}(\{A, A^{-1}\}, SO(m)) \leq C(\|\Omega\|_{L^2} + \|\Omega_1\|_{L^{2,1}} + \|Q\|_{\dot{H}^{1/2}}) \quad (69)$$

and

$$(-\Delta)^{1/4}[Av] = \mathcal{J}(B, v) + Ah_r + Ah_R, \quad (70)$$

with $\mathcal{J}(B, v)$ satisfying (63). We write $v = v_1 + v_2 + v_3 + v_4$ where

$$(-\Delta)^{1/4}[Av_1] = \mathcal{J}(B, v); \quad (71)$$

$$(-\Delta)^{1/4}[Av_2] = Ah_r - \left(\int_{B(0,r)} Ah_r dx \right) \delta_0; \quad (72)$$

$$(-\Delta)^{1/4}[Av_3] = Ah_R, \quad (73)$$

$$(-\Delta)^{1/4}[Av_4] = \left(\int_{B(0,r)} Ah_r dx \right) \delta_0; \quad (74)$$

Let $\Lambda > 2$ be such that $\Lambda r < (2\Lambda)^{-1}R$. We are going to estimate the $L^{2,1}(B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r))$ norm of v_1, v_2, v_3 .

1. Estimate of v_1 .

By the properties of \mathcal{J} we have

$$\|Av_1\|_{L^{2,1}(\mathbb{R})} \leq C\|(-\Delta)^{-1/4}[J(B, v)]\|_{L^{2,1}(\mathbb{R})} \quad (75)$$

$$\leq C\|B\|_{\dot{H}^{1/2}}\|v\|_{L^2}. \quad (76)$$

2. Estimate of v_2 .

We set $\phi_r := Ah_r - \left(\int_{B(0,r)} Ah_r dx\right) \delta_0$ and take $x \in B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r)$. By using the fact that $\int_{B(0,r)} \phi_r(y) dy = 0$, we get

$$\begin{aligned} Av_2(x) &= \int_{B(0,r)} \phi_r(y) \frac{1}{|x-y|^{1/2}} dy = \frac{1}{|x|^{1/2}} \sum_{k=0}^{+\infty} \frac{c_k}{|x|^k} \int_{B(0,r)} \phi_r(y) y^k dy \\ &= \sum_{k=1}^{+\infty} \frac{c_k}{|x|^{k+1/2}} \int_{B(0,r)} \phi_r(y) y^k dy, \end{aligned}$$

where the $c_k = \frac{1 \times 3 \times \dots (2k-1)}{2 \times 4 \times \dots \times 2k}$.

We observe that for every $k \geq 1$, $\frac{1}{|x|^{k+1/2}} \in L^{2,1}(B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r))$ for all $\Lambda > 1$ such that $\Lambda r < (2\Lambda)^{-1}R$, with

$$\left\| \frac{1}{|x|^{k+1/2}} \right\|_{L^{2,1}(B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r))} \lesssim \left(\frac{1}{\Lambda r} \right)^k.$$

Therefore

$$\begin{aligned} \|v_2\|_{L^{2,1}(B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r))} &\leq C\|\phi_r\|_{L^1} \sum_{k=1}^{+\infty} \frac{c_k}{(\Lambda r)^k} r^k \\ &\leq C\|\phi_r\|_{L^1} \sum_{k=1}^{+\infty} \frac{c_k}{\Lambda^k} < +\infty \\ &\leq C\|\Omega\|_{L^2}\|v\|_{L^2} \left(\sum_{k=1}^{+\infty} \frac{c_k}{\Lambda^k} < +\infty \right). \end{aligned}$$

3. Estimate of v_3 .

Since Ah_R has support in $B^c(0, R)$ then by interpolation one gets that $Av_3 \in L^{2,1}(B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r))$ and

$$\|Av_3\|_{L^{2,1}(B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r))} \leq C\|h_R\|_{L^1} \leq C\|\Omega\|_{L^2}\|v\|_{L^2}.$$

3. Estimate of v_4 .

We have

$$v_4(x) = A^{-1}(x) \vec{c}_r \frac{1}{|x|^{1/2}},$$

where

$$\vec{c}_r = \int_{B(0,r)} A h_r dx.$$

Since $A^{-1} \in \dot{H}^{1/2}(\mathbb{R})$, it verifies the following estimate (see lemma 33.1 of [16]):

$$\left\| \frac{A^{-1}(x) - A^{-1}(-x)}{|x|^{1/2}} \right\|_{L^2(\mathbb{R})} \leq C \|A^{-1}\|_{\dot{H}^{1/2}} \quad (77)$$

Therefore by combining the $L^{2,1}$ estimates on $B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r)$ of v_1, v_2, v_3 we obtain

$$v(x) = (A^{-1}(x))^+ \vec{c}_r \frac{1}{|x|^{1/2}} + (A^{-1}(x))^- \vec{c}_r \frac{1}{|x|^{1/2}} + h(x) \quad (78)$$

with $h(x) = v_1 + v_2 + v_3$ and $\|h\|_{L^{2,1}(B(0, (2\Lambda)^{-1}R) \setminus B(0, \Lambda r))} \leq C \|v\|_{L^2(\mathbb{R})}$, $\vec{c}_r = \int_{B(0,r)} A(\Omega^r v) dx$. Finally from (77) it follows that $g(x) = \text{asymm}(A^{-1}(x)) \vec{c}_r \frac{1}{|x|^{1/2}}$ satisfies $g \in L^2(\mathbb{R})$ with $\|g\|_{L^2(\mathbb{R})} \leq C |\vec{c}_r|$. We can conclude the proof. \square

Lemma 2.1 (ε -regularity) *Let $v \in L^2(\mathbb{R}, \mathbb{R}^m)$ be a solution of*

$$(-\Delta)^{1/4} v = \Omega v + \Omega^1 v + \mathcal{Z}(Q, v). \quad (79)$$

Then there exists $\varepsilon_0 > 0$ such that if

$$\sum_{j \geq 0} 2^{-j/2} (\|\Omega\|_{L^2(B(x, 2^j r))} + \|\Omega^1\|_{L^{2,1}(B(x, 2^j r))} + \|(-\Delta)^{1/4} Q\|_{L^2(B(x, 2^j r))}) \leq \varepsilon_0, \quad (80)$$

then there is $p > 2$ (independent of u) such that for every $y \in B(x, r/2)$ we have

$$\left(r^{p/2-1} \int_{B(y, r/2)} |v|^p dx \right)^{1/p} \leq C \sum_{j \geq 0} 2^{-j/2} (\|\Omega\|_{L^2(B(x, 2^j r))} + \|\Omega^1\|_{L^{2,1}(B(x, 2^j r))} + \|(-\Delta)^{1/4} Q\|_{L^2(B(x, 2^j r))}), \quad (81)$$

where $C > 0$ depends on $\|v\|_{L^2}$.

Finally we have

Lemma 2.2 ($L^{2,\infty}$ estimates) *There exists $\delta > 0$ such that for any $v \in L^2(\mathbb{R})$ solution of (51) and $\lambda, \Lambda > 0$ with $\lambda < (2\Lambda)^{-1}$ satisfying*

$$\sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left[\|\Omega\|_{L^2(B(0, 2\rho) \setminus B(0, \rho))} + \|(-\Delta)^{1/4} Q\|_{L^2(B(0, 2\rho) \setminus B(0, \rho))} + \|\Omega^1\|_{L^{2,1}(B(0, 2\rho) \setminus B(0, \rho))} \right] \leq \delta$$

then

$$\begin{aligned} \|v\|_{L^{2,\infty}(B(0,(2\Lambda)^{-1}) \setminus B(0,\lambda))} &\leq C \sup_{\rho \in [\lambda, (2\Lambda)^{-1}]} \left[\|\Omega\|_{L^2(B(0,2\rho) \setminus B(0,\rho))} \right. \\ &\quad \left. + \|(-\Delta)^{1/4} Q\|_{L^2(B(0,2\rho) \setminus B(0,\rho))} + \|\Omega^1\|_{L^{2,1}(B(0,2\rho) \setminus B(0,\rho))} \right]. \end{aligned} \quad (82)$$

where C is independent of λ, Λ .

The proof of Lemmae 2.1 and 2.2 are the same of Lemma 2.1 and Lemma 3.2 in [4] and therefore we omit them.

We show now a point removability type result for solution of (51).

Proposition 2.2 [Point removability] *Let $v \in L^2(\mathbb{R})$ be a solution of (51) in $\mathcal{D}'(\mathbb{R} \setminus \{a_1, \dots, a_\ell\})$. Then it is a solution of (51) $\mathcal{D}'(\mathbb{R})$.*

Proof of Proposition 2.2. The fact that

$$(-\Delta)^{1/4} v = \Omega v + \Omega^1 v + \mathcal{Z}(Q, v), \quad \text{in } \mathcal{D}'(\mathbb{R} \setminus \{a_1, \dots, a_\ell\})$$

implies that the distribution $\phi := (-\Delta)^{1/4} v - \Omega v - \Omega^1 v - \mathcal{Z}(Q, v)$ is of order $p = 1$ and supported in $\{a_1, \dots, a_\ell\}$. Therefore by Schwartz Theorem (see 6.1.5. of [1]) one has

$$\phi = \sum_{|\alpha| \leq 1} c_\alpha \partial^\alpha \delta_{a_i}.$$

Since $\phi \in \dot{L}^1(\mathbb{R})$, then the above implies that $c_\alpha = 0$ and thus v is a solution of (51) in $\mathcal{D}'(\mathbb{R})$. We conclude the proof of Proposition 2.2. \square

3 Pohozaev Identities

In this section we provide some new Pohozaev identities in \mathbb{R} and S^1 which are in particular satisfied by horizontal 1/2 harmonic in $\mathfrak{H}^{1/2}(\mathbb{R})$.

3.1 Pohozaev Identities for $(-\Delta)^{1/2}$ in \mathbb{R}

We first determine the solution of

$$\begin{cases} \partial_t G + (-\Delta)^{1/2} G = 0 & t > 0 \\ G(0, x) = \delta & t = 0. \end{cases} \quad (83)$$

In order to solve (83) we consider the Fourier Transform of G with respect to x

$$\hat{G}(t, \xi) = \int_{\mathbb{R}} G(t, x) e^{-ix\xi} dx$$

The function \hat{G} satisfies the problem

$$\begin{cases} \partial_t \hat{G} + |\xi| \hat{G} = 0 & x \in \mathbb{R}, t > 0 \\ \hat{G}(0, x) = \hat{\delta} = 1 & x \in \mathbb{R}, t = 0. \end{cases} \quad (84)$$

The solution of (84) is $\hat{G}(t, \xi) = e^{-t|\xi|}$. Then

$$G(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{G}(t, \xi) e^{ix\xi} d\xi = \frac{1}{\pi} \frac{t}{x^2 + t^2}.$$

The following equalities hold

$$\partial_t G = \frac{1}{\pi} \frac{x^2 - t^2}{(t^2 + x^2)^2}, \quad \partial_x G = -\frac{1}{\pi} \frac{2xt}{(t^2 + x^2)^2}.$$

Theorem 3.1 [Case on \mathbb{R}] Let $u \in \dot{H}_{loc}^{1/2}(\mathbb{R}, \mathbb{R}^m)$ such that

$$\left\langle \frac{du}{dx}, (-\Delta)^{1/2} u \right\rangle = 0 \quad a.e \text{ in } \mathbb{R}. \quad (85)$$

Assume that

$$\int_{\mathbb{R}} |u - u_0| dx < +\infty, \quad \int_{\mathbb{R}} \left| \frac{du}{dx}(x) \right| dx < +\infty.. \quad (86)$$

Then the following identity holds

$$\left| \int_{x \in \mathbb{R}} \frac{x^2 - t^2}{(x^2 + t^2)^2} u(x) dx \right|^2 = \left| \int_{x \in \mathbb{R}} \frac{2xt}{(x^2 + t^2)^2} u(x) dx \right|^2. \quad (87)$$

Proof. We assume for simplicity that $u_0 = 0$. Set $G_t(x) := G(t, x)$, multiply the equation (85) by $xG_t(x)$ and we integrate. For every $j \in \{1, \dots, m\}$ we get by Plancherel Theorem

$$\begin{aligned} 0 &= \int_{\mathbb{R}} x G_t \frac{du^j}{dx} \overline{(-\Delta)^{1/2} u^j} dx = \int_{\mathbb{R}} \mathcal{F} \left[x G_t \frac{du^j}{dx} \right] \overline{\mathcal{F}[(-\Delta)^{1/2} u^j]} dx \\ &= \int_{\mathbb{R}} \mathcal{F}[G_t] * \mathcal{F} \left[x \frac{du^j}{dx} \right] |\xi| \overline{\mathcal{F}[u^j]} d\xi \\ &= \int_{\mathbb{R}} \mathcal{F}[G_t] * i \frac{d}{d\xi} \mathcal{F} \left[\frac{du^j}{dx} \right] |\xi| \overline{\mathcal{F}[u^j]} d\xi \\ &= \int_{\mathbb{R}} \mathcal{F}[G_t] * i \frac{d}{d\xi} (\xi \mathcal{F}[u^j]) |\xi| \overline{\mathcal{F}[u^j]} d\xi \\ &= - \iint_{\mathbb{R}^2} \hat{G}_t(\xi - \eta) \frac{\partial}{\partial \eta} (\eta \hat{u}^j(\eta)) |\xi| \overline{\hat{u}^j}(\xi) d\xi d\eta \\ &= \iint_{\mathbb{R}^2} \frac{\partial}{\partial \eta} [\hat{G}_t(\xi - \eta)] \eta |\xi| \hat{u}^j(\eta) \overline{\hat{u}^j}(\xi) d\xi d\eta \\ &= \iint_{\mathbb{R}^2} t e^{-t|\xi - \eta|} \frac{\xi - \eta}{|\xi - \eta|} \eta |\xi| \hat{u}^j(\eta) \overline{\hat{u}^j}(\xi) d\xi d\eta \end{aligned} \quad (88)$$

We symmetrize (88)

$$0 = \iint_{\mathbb{R}^2} t e^{-t|\xi+\eta|} \frac{-\xi+\eta}{|-\xi+\eta|} \xi(|\eta|) \hat{u}^j(\xi) \overline{\hat{u}^j(\eta)} d\xi d\eta \quad (89)$$

Then we sum (88) and (89) and we take the real part:

$$\begin{aligned} 0 &= \iint_{\mathbb{R}^2} t e^{-t|\xi-\eta|} \frac{\xi-\eta}{|\xi-\eta|} [\eta|\xi| - \xi|\eta|] \Re \left[\hat{u}^j(\xi) \overline{\hat{u}^j(\eta)} \right] d\xi d\eta \\ &= \iint_{\mathbb{R}^2} t e^{-t|\xi-\eta|} \frac{\xi-\eta}{|\xi-\eta|} [\eta|\xi| - \xi|\eta|] \left[a^j(\xi) a^j(\eta) + b^j(\xi) b^j(\eta) \right] d\xi d\eta \end{aligned} \quad (90)$$

where for every j we set $\hat{u}^j(\eta) = a^j(\eta) + i b^j(\eta)$.

We write

$$\begin{aligned} a^{j,\pm}(\eta) &= \frac{a^j(\eta) \pm a^j(-\eta)}{2} \\ b^{j,\pm}(\eta) &= \frac{b^j(\eta) \pm b^j(-\eta)}{2}. \end{aligned}$$

We observe that $a^{j,+}, b^{j,+}$ are even and $a^{j,-}, b^{j,-}$ are odd. Moreover since u^j is real we also have $b^{j,+} = 0, a^{j,-} = 0, b^{j,-} = b, a^{j,-} = a$. We can also write

$$\begin{aligned} a^{j,+}(\xi) &= \int_{\mathbb{R}} \frac{u^j(x) + u^j(-x)}{2} \cos(x\xi) dx - i \int_{\mathbb{R}} \frac{u^j(x) + u^j(-x)}{2} \sin(x\xi) dx \\ &= \mathcal{F}(u^{j,+})(\xi); \end{aligned} \quad (91)$$

$$\begin{aligned} b^{j,-}(\xi) &= i \left\{ \int_{\mathbb{R}} \frac{u^j(x) - u^j(-x)}{2} \cos(x\xi) dx - i \int_{\mathbb{R}} \frac{u^j(x) - u^j(-x)}{2} \sin(x\xi) dx \right\} \\ &= i \mathcal{F}(u^{j,-})(\xi) \end{aligned} \quad (92)$$

We set

$$Q(\alpha, \beta) := \iint_{\mathbb{R}^2} \frac{\xi-\eta}{|\xi-\eta|} t e^{-t(|\xi-\eta|)} [\eta|\xi| - \xi|\eta|] \alpha(\eta) \beta(\xi) d\xi d\eta.$$

It follows that

$$0 = Q(a^{j,+}, a^{j,+}) + Q(b^{j,-}, b^{j,-}). \quad (93)$$

Estimate of $Q(a^{j,+}, a^{j,+})$

$$\begin{aligned}
Q(a^{j,+}, a^{j,+}) &= \iint_{\mathbb{R}^2} \frac{\xi - \eta}{|\xi - \eta|} t e^{-t(|\xi - \eta|)} [\eta |\xi| - \xi |\eta|] a^{j,+}(\eta) a^{j,+}(\xi) d\xi d\eta \\
&= 2 \int_{\xi > 0} \int_{\eta < 0} \eta \xi \frac{\xi - \eta}{|\xi - \eta|} t e^{-t(|\xi - \eta|)} a^{j,+}(\eta) a^{j,+}(\xi) d\xi d\eta \\
&\quad - 2 \int_{\xi < 0} \int_{\eta > 0} \eta \xi \frac{\xi - \eta}{|\xi - \eta|} t e^{-t(|\xi - \eta|)} a^{j,+}(\eta) a^{j,+}(\xi) d\xi d\eta \\
&= 4 \int_{\xi > 0} \int_{\eta < 0} \eta \xi t e^{-t(|\xi - \eta|)} a^{j,+}(\eta) a^{j,+}(\xi) d\xi d\eta \\
&= -4t \left(\int_{\xi > 0} \xi e^{-t\xi} a^{j,+}(\xi) d\xi \right)^2 \\
&= -4t \left(\int_{\xi > 0} \xi e^{-t\xi} \mathcal{F}(u^{j,+})(\xi) d\xi \right)^2 = -4t \left(\frac{1}{2} \int_{x \in \mathbb{R}} |\xi| e^{-t|\xi|} \mathcal{F}(u^{j,+})(\xi) d\xi \right)^2 \\
&= -t \left(\int_{\xi \in \mathbb{R}} \mathcal{F}[(-\Delta)^{1/2} G](\xi) \mathcal{F}(u^{j,+})(\xi) d\xi \right)^2 \\
&= -t \left(\int_{x \in \mathbb{R}} \partial_t G(x, t) (u^{j,+})(x) dx \right)^2 \\
&= t \frac{1}{\pi^2} \left(\int_{x \in \mathbb{R}} \frac{x^2 - t^2}{(x^2 + t^2)^2} (u^{j,+})(x) dx \right)^2
\end{aligned} \tag{94}$$

Estimate of $Q(b^{j,-}, b^{j,-})$

$$\begin{aligned}
Q(b^{j,-}, b^{j,-}) &= \iint_{\mathbb{R}^2} \frac{\xi - \eta}{|\xi - \eta|} t e^{-t(|\xi - \eta|)} [\eta |\xi| - \xi |\eta|] b^{j,-}(\eta) b^{j,-}(\xi) d\xi d\eta \\
&= 2 \int_{\xi > 0} \int_{\eta < 0} \eta \xi \frac{\xi - \eta}{|\xi - \eta|} t e^{-t(|\xi - \eta|)} b^{j,-}(\eta) b^{j,-}(\xi) d\xi d\eta \\
&\quad - 2 \int_{\xi < 0} \int_{\eta > 0} \eta \xi \frac{\xi - \eta}{|\xi - \eta|} t e^{-t(|\xi - \eta|)} b^{j,-}(\eta) b^{j,-}(\xi) d\xi d\eta \\
&= 4 \int_{\xi > 0} \int_{\eta < 0} \eta \xi t e^{-t(|\xi - \eta|)} b^{j,-}(\eta) b^{j,-}(\xi) d\xi d\eta \\
&= 4t \left(\int_{\xi > 0} \xi e^{-t\xi} b^{j,-}(\xi) d\xi \right)^2 \\
&= 4t \left(\int_{\xi > 0} \xi e^{-t\xi} i \mathcal{F}(u^{j,-})(\xi) d\xi \right)^2 = 4t \left(\frac{1}{2} \int_{\xi \in \mathbb{R}} i \xi e^{-t|\xi|} \mathcal{F}(u^{j,-})(\xi) d\xi \right)^2 \\
&= t \left(\int_{\xi \in \mathbb{R}} \mathcal{F}[\partial_x G](\xi) \mathcal{F}(u^{j,-})(\xi) d\xi \right)^2 \\
&= t \left(\int_{x \in \mathbb{R}} \partial_x G(x, t) (u^{j,-})(x) dx \right)^2 \\
&= t \frac{1}{\pi^2} \left(\int_{x \in \mathbb{R}} \frac{2xt}{(x^2 + t^2)^2} (u^{j,-})(x) dx \right)^2
\end{aligned} \tag{95}$$

From (93), (94) and (95) it follows that

$$\left| \int_{x \in \mathbb{R}} \frac{x^2 - t^2}{(x^2 + t^2)^2} (u^+)(x) dx \right|^2 = \left| \int_{x \in \mathbb{R}} \frac{2xt}{(x^2 + t^2)^2} (u^-)(x) dx \right|^2 \tag{96}$$

In particular we get

$$\left| \int_{x \in \mathbb{R}} \frac{x^2 - t^2}{(x^2 + t^2)^2} u(x) dx \right| = \left| \int_{x \in \mathbb{R}} \frac{2xt}{(x^2 + t^2)^2} u(x) dx \right|, \tag{97}$$

which achieves the proof of theorem 3.1. \square

Preliminary computations

We observe that

$$\frac{x^2 - t^2}{(x^2 + t^2)^2} = \frac{d}{dx} \left(\frac{-x}{x^2 + t^2} \right), \quad \frac{2xt}{(x^2 + t^2)^2} = \frac{d}{dx} \left(\frac{-t}{x^2 + t^2} \right).$$

Computation of $\mathcal{F} \left[\frac{-1}{x^2+1} \right]$ and $\mathcal{F} \left[\frac{-x}{x^2+1} \right]$.

$$\begin{aligned}\mathcal{F}\left[\frac{-1}{x^2+1}\right](\xi) &= \int_{\mathbb{R}} \frac{-1}{x^2+1} e^{-ix\xi} d\xi \\ &= -\pi e^{-|\xi|}.\end{aligned}$$

$$\mathcal{F}\left[\frac{-x}{x^2+1}\right](\xi) = i \frac{d}{d\xi} \left(\mathcal{F}\left[\frac{-1}{x^2+1}\right] \right) = -i\pi \frac{\xi}{|\xi|} e^{-|\xi|}.$$

Computation of $(-\Delta)^{-1/4}[\frac{x^2-1}{(1+x^2)^2}]$ and $(-\Delta)^{-1/4}[\frac{2x}{(1+x^2)^2}]$

$$\begin{aligned}(-\Delta)^{-1/4} \left[\frac{x^2-1}{(1+x^2)^2} \right] &= \mathcal{F}^{-1} \left[|\xi|^{-1/2} \mathcal{F} \left[\frac{d}{dx} \left(\frac{-x}{x^2+1} \right) \right] \right] \\ &= \mathcal{F}^{-1} [|\xi|^{-1/2} i \xi \mathcal{F} \left[\frac{-x}{x^2+1} \right]] \\ &= \pi \mathcal{F}^{-1} \left[|\xi|^{-1/2} \xi \frac{\xi}{|\xi|} e^{-|\xi|} \right] = \pi \mathcal{F}^{-1} [|\xi|^{1/2} e^{-|\xi|}] \quad (98) \\ &= \pi \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{1/2} e^{-|\xi|} e^{-ix\xi} d\xi = \sqrt{\pi} \Re \left(\frac{1}{(1+ix)^{3/2}} \right) \\ &= \sqrt{\pi} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right).\end{aligned}$$

$$\begin{aligned}(-\Delta)^{-1/4} \left[\frac{2x}{(1+x^2)^2} \right] &= \mathcal{F}^{-1} \left[|\xi|^{-1/2} \mathcal{F} \left[\frac{d}{dx} \left(\frac{-1}{x^2+1} \right) \right] \right] \\ &= \mathcal{F}^{-1} [|\xi|^{-1/2} i \xi \mathcal{F} \left[\frac{-1}{x^2+1} \right]] \\ &= \pi i \mathcal{F}^{-1} [|\xi|^{-1/2} \xi e^{-|\xi|}] = \pi i \frac{1}{2\pi} \int_{\mathbb{R}} |\xi|^{-1/2} \xi e^{-|\xi|} e^{-ix\xi} d\xi \quad (99) \\ &= \sqrt{\pi} i \Im \left(\frac{1}{(1+ix)^{3/2}} \right) = \sqrt{\pi} \left(\frac{\sin(\arctan(-x))}{(1+x^2)^{3/4}} \right).\end{aligned}$$

Next we define the following operators:

$$M^-[w](t) := \int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\sin(\arctan(-x))}{(1+x^2)^{3/4}} \right) w(tx) dx, \quad (100)$$

$$M^+[w](t) := \int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right) w(tx) dx, \quad (101)$$

Proposition 3.1 *The operators M^+ (resp. M^-) is an isomorphism from L_+^p to L_+^p (resp. L_-^p to L_-^p) for every $p \in (1, +\infty)$.*

Proof. We prove the proposition for M^+ , it is exactly the same for M^- .

Claim 1. $M^+ : L_+^p(\mathbb{R}) \rightarrow L_+^p(\mathbb{R})$, for every $p \in (1, +\infty)$.

Proof of the claim 1. Let $p \in (1, \infty)$ and $p' = \frac{p}{p-1}$ be the conjugate of p .

Let us introduce $g(x) := \frac{(1+x^2)^\beta}{|x|^\alpha}$ where $\beta = \alpha = \frac{1}{4p'}$, and let $w \in L^p(\mathbb{R})$.

$$\begin{aligned}
\|M^+[w](t)\|_{L^p(\mathbb{R}^+)}^p &= (\pi)^{p/2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right) w(tx) dx \right)^p dt \\
&\leq (\pi)^{p/2} \left(\int_{\mathbb{R}} |w(y)|^p dy \right) \left(\int_{\mathbb{R}} \left(\frac{g(x)}{(1+x^2)^{3/4}} \right)^{p'} dx \right)^{p/p'} \left(\int_{\mathbb{R}} \frac{1}{|x|g^p(x)} dx \right) \\
&= (\pi)^{p/2} \left(\int_{\mathbb{R}} |w(y)|^p dy \right) \\
&\quad \left(\int_{\mathbb{R}} |x|^{-1/4} (1+x^2)^{-\frac{3}{4}p'+1/4} dx \right)^{p/p'} \left(\int_{\mathbb{R}} |x|^{-1+\frac{p}{4p'}} (1+x^2)^{-\frac{p}{4p'}} dx \right) \\
&\leq C_p \|w\|_{L^p(\mathbb{R})}^p. \quad \square
\end{aligned}$$

Claim 2: The adjoint of M^+ in L_+^2 is $\mathcal{F}^{-1} \circ M^+ \circ \mathcal{F}^{-1}$

Proof of claim 2. Thanks to (98), we have

$$\pi |\xi|^{-1/2} \xi \frac{\xi}{|\xi|} e^{-|\xi|} = \mathcal{F} \left(\sqrt{\pi} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right) \right).$$

Then let $w \in L^2(\mathbb{R})$, for $t \neq 0$,

$$M^+[w](t) = \pi \int_{\mathbb{R}} |\xi|^{1/2} e^{-|\xi|} \mathcal{F}[w(tx)](\xi) d\xi = \pi \int_{\mathbb{R}} |\xi|^{1/2} |t|^{-1} e^{-|\xi|} \hat{w} \left(\frac{\xi}{t} \right) d\xi.$$

Here we used the fact that \hat{w} is real. Then, let $w, v \in L_+^2$, we have

$$\begin{aligned}
\langle v, M^+[w] \rangle &= \pi \int_{\mathbb{R}} v(t) \left(\int_{\mathbb{R}} |\xi|^{1/2} e^{-|\xi|} |t|^{-1} \hat{w} \left(\frac{\xi}{t} \right) d\xi \right) dt \\
&= \pi \int_{\mathbb{R}} \int_{\mathbb{R}} |\xi|^{1/2} e^{-|\xi|} v(t) |t|^{-1} \hat{w} \left(\frac{\xi}{t} \right) dt d\xi \\
&= \pi \int_{\mathbb{R}} \int_{\mathbb{R}} |y|^{1/2} e^{-|ty|} v(t) |t|^{1/2} \hat{w}(y) dt dy \\
&= \pi \int_{\mathbb{R}} \int_{\mathbb{R}} |x|^{\frac{1}{2}} e^{-|x|} |y|^{-1} v \left(\frac{x}{y} \right) \hat{w}(y) dx dy \\
&= \langle M^+ [\mathcal{F}^{-1}(v)], \mathcal{F}(w) \rangle \\
&= \langle \mathcal{F}^{-1} (M^+ [\mathcal{F}^{-1}(v)]), w \rangle
\end{aligned}$$

Claim 3: $\text{Ker}(M^+) = \text{Ker}(M^+)^* = \{0\}$

Proof of claim 3. We observe that $M^+[w] = 0$ if and only if the Laplace transform of $|\xi|^{-1/2} \xi \hat{w}(\xi)$ is zero. Here we use once more that w is even. Since the Laplace transform is injective, it implies that $|\xi|^{-1/2} \xi \hat{w}(\xi) = 0$ a.e, namely $w = 0$ a.e. The same hold for the adjoint.

By combining Claim 1, 2, 3 we deduce that M^+ is bijective bounded linear operators from L_+^2 to L_+^2 . The Open Mapping Theorem implies that it is an isomorphism as well.

By density arguments we get that M^+ is also an isomorphism from L_+^p to L_+^p for every $p \in (1, \infty)$ and therefore from $L_+^{2,1}$ to $L_+^{2,1}$ too. We conclude the proof of Proposition 3.1. \square .

3.2 Pohozaev Identities for $(-\Delta)^{1/2}$ in S^1

In this section we will derive a Pohozaev formula for C^1 , stationary 1/2-harmonic on S^1 , namely satisfying

$$\frac{d}{da} \int_{S^1} |(-\Delta)^{1/4} (u \circ \phi_a)|^2|_{a=0} = 0 \quad (102)$$

where $\phi_a: S^1 \rightarrow S^1$ is a family of diffeomorphisms such that $\phi_0 = Id$.

Stationary 1/2-harmonic maps satisfy

$$\begin{aligned}
0 &= (-\Delta)^{1/2} (u \circ \phi_a) \cdot \frac{d}{d\theta} (u \circ \phi_a)|_{a=0} \\
&= (-\Delta)^{1/2} (u \circ \phi_a) \cdot \frac{du}{d\phi} \circ \phi_a \frac{d\phi_a}{d\theta} |_{a=0} \quad \text{in } \mathcal{D}'
\end{aligned} \quad (103)$$

If we choose for every $a \in [0, \frac{1}{2})$, $\phi_a = \frac{e^{i\theta} - a}{1 - ae^{i\theta}}$ and we set $u_a(\theta) = u(\frac{e^{i\theta} - a}{1 - ae^{i\theta}})$ (namely we consider the composition of u with a family of Möbius transformations of the disk).

In this case we get

$$\frac{du}{d\phi} \circ \phi_a|_{a=0} = -ie^{-i\theta} \partial_\theta u_0(\theta), \quad \frac{d\phi_a}{d\theta}|_{a=0} = -1 + e^{2i\theta}. \quad (104)$$

Moreover if ϕ_a is conform we have

$$(-\Delta)^{1/2}(u \circ \phi_a) = (-\Delta)^{1/2}u_a = e^{\lambda_a}((-\Delta)^{1/2}u) \circ \phi_a, \quad (105)$$

where $\lambda_a = \log(|\frac{\partial \phi_a}{\partial \theta}(\theta)|)$. Therefore

$$\begin{aligned} 0 &= (-\Delta)^{1/2}u \circ \phi_a \cdot \frac{du}{d\phi} \circ \phi_a \frac{d\phi_a}{d\theta}|_{t=0} \text{ in } \mathcal{D}' \\ &= (-\Delta)^{1/2}u(e^{i\theta}) \cdot (-ie^{-i\theta} \partial_\theta u_0(\theta)(-1 + e^{2i\theta})) \\ &= (-\Delta)^{1/2}u_0 \cdot (2 \sin(\theta) \partial_\theta u_0(\theta)). \end{aligned} \quad (106)$$

We also observe that the energy $\int_{S^1} |(-\Delta)^{1/4}u|^2 d\theta$ is invariant with respect to the trace of Möbius transformations of the disk and therefore for every function $w \in \dot{H}^{1/2}(S^1)$ we have

$$\frac{d}{da} \int_{S^1} |(-\Delta)^{1/4}w_a|^2 d\theta = \frac{d}{da} \int_{S^1} |(-\Delta)^{1/4}w|^2 d\theta = 0.$$

In particular we get

$$2 \int_{S^1} \frac{dw_a}{da}(\theta)|_{a=0} (-\Delta)^{1/2}w d\theta = 4 \int_{S^1} \sin(\theta) \partial_\theta w (-\Delta)^{1/2}w d\theta = 0.$$

In the sequel we identify S^1 with $[-\pi, \pi]$. We consider the following problem

$$\begin{cases} \partial_t F + (-\Delta)^{1/2}F = 0 & \theta \in [-\pi, \pi], t > 0 \\ F(0, \theta) = \delta_0(x) & \theta \in [-\pi, \pi]. \end{cases} \quad (107)$$

Claim: The solution of (107) is given by

$$F(\theta, t) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{-t|n|} e^{in\theta} = \frac{e^{2t} - 1}{e^{2t} - 2e^t \cos(\theta) + 1}.$$

Proof of the Claim. For every $t > 0$ we write

$$F(\theta, t) = \sum_{n=-\infty}^{+\infty} F_n(t) e^{in\theta}. \quad (108)$$

By plugging the formula into the equation we get for every $n \in \mathbb{Z}$:

$$\frac{d}{dt}F_n(t) + |n|F_n(t) = 0.$$

Therefore $F_n(t) = C_n e^{-t|n|}$ and $F(\theta, t) = \sum_{n=-\infty}^{+\infty} C_n e^{-t|n|} e^{in\theta}$. Since $F(\theta, 0) = \delta_0 = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in\theta}$, namely $F(\theta, 0) * f(\theta) = f(\theta)$ for every distribution f , we get that $C_n = \int_{S^1} F(\theta, 0) e^{-in\theta} d\theta = \frac{1}{2\pi}$ for every $n \in \mathbb{Z}$. Hence F is as in (108) and the claim is proved. \square

Theorem 3.2 [Case on S^1] Let $u \in \dot{H}_{loc}^{1/2}(S^1, \mathbb{R}^m)$ such that

$$\frac{\partial u}{\partial \theta} \cdot (-\Delta)^{1/2} u = 0 \quad a.e \text{ in } S^1. \quad (109)$$

Then the following identity holds

$$\left| \int_{S^1} u(z) \partial_t F(z) d\theta \right|^2 = \left| \int_{S^1} u(z) \partial_\theta F(z) d\theta \right|^2 \quad (110)$$

Proof of Theorem 3.2

Now we write $u(\theta) = \sum_{n \in \mathbb{Z}} u_n e^{in\theta}$. Then the following equalities hold:

$$\begin{aligned} \sin(\theta) \partial_\theta u &= \frac{1}{2} \sum_{n \in \mathbb{Z}} (e^{i\theta} - e^{-i\theta}) n u_n e^{in\theta} \\ &= \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{in\theta} ((n-1)u_{n-1} - (n+1)u_{n+1}) \\ \overline{(-\Delta)^{1/2} u} &= \sum_{n \in \mathbb{Z}} |n| \bar{u}_n e^{-in\theta} \\ \sin(\theta) \partial_\theta \overline{(-\Delta)^{1/2} u} &= \sum_{n, m \in \mathbb{Z}} e^{i(n-m)\theta} |m| [(n-1)u_{n-1} - (n+1)u_{n+1}] \bar{u}_m. \end{aligned} \quad (111)$$

By combining (106) and (111) we get

$$\begin{aligned} 0 &= \sum_{n, m \in \mathbb{Z}} |m| e^{-t|n-m|} [(n-1)u_{n-1} - (n+1)u_{n+1}] \bar{u}_m \\ &= \sum_{n, m \in \mathbb{Z}} |m| n u_n \bar{u}_m [e^{t|n-m+1|} - e^{-t|n-m-1|}]. \end{aligned} \quad (112)$$

We first symmetrize (112). We get first

$$0 = - \sum_{n, m \in \mathbb{Z}} |n| m u_m \bar{u}_n [e^{t|n-m+1|} - e^{-t|n-m-1|}]. \quad (113)$$

We sum (112) and (113)

$$0 = \sum_{n,m \in \mathbb{Z}} | [e^{t|n-m+1|} - e^{-t|n-m-1|}] [|m|nu_n\bar{u}_m - |n|m\bar{u}_nu_m]. \quad (114)$$

For every $n \in \mathbb{Z}$ we set

$$\begin{aligned} u_n &= a_n + ib_n \\ a_n^+ &= \frac{a_n + a_{-n}}{2}, \quad a_n^- = \frac{a_n - a_{-n}}{2} \\ b_n^+ &= \frac{b_n + b_{-n}}{2}, \quad b_n^- = \frac{b_n - b_{-n}}{2}. \end{aligned}$$

We observe that since the components of u are real we have $a_n^- = 0$ and $b_n^+ = 0$, $a_n^+ = u_n^+$, $b_n^- = iu_n^-$. We take the real part of (114) and get

$$\begin{aligned} 0 &= \sum_{n,m \in \mathbb{Z}} (|m|n - |n|m) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [a_m a_n + b_m b_n] \\ &= \sum_{n,m \in \mathbb{Z}} (|m|n - |n|m) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [a_m^+ a_n^+] \end{aligned} \quad (115)$$

$$+ \sum_{n,m \in \mathbb{Z}} (|m|n - |n|m) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [b_m^- b_n^-]. \quad (116)$$

Estimate of (115)

$$\begin{aligned}
& \sum_{n,m \in \mathbb{Z}} (|m|n - |n|m) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [a_m^+ a_n^+] \\
= & \sum_{n>0, m<0} (-2mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [a_m^+ a_n^+] \\
& + \sum_{n<0, m>0} (2mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [a_m^+ a_n^+] \\
= & \sum_{n>0, m<0} (-2mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [a_m^+ a_n^+] \\
& + \sum_{n>0, m<0} (2(-m)(-n)) [e^{-t|-n+m+1|} - e^{-t|-n+m-1|}] [a_{-m}^+ a_{-n}^+] \\
= & -4 \sum_{n>0, m<0} (mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [a_m^+ a_n^+] \\
= & 4 \sum_{n>0, m>0} (-m)n [e^{-t|n+m+1|} - e^{-t|n+m-1|}] [a_{-m}^+ a_n^+] \\
= & -4 \sum_{n>0, m>0} mn [e^{-t(n+m+1)} - e^{-t(n+m-1)}] [a_m^+ a_n^+] \\
= & -4 (e^{-t} - e^t) \left(\sum_{n>0} n e^{-tn} a_n^+ \right)^2 = 8 \sinh(t) \left(\sum_{n \in \mathbb{Z}} |n| e^{-t|n|} u_n^+ \right)^2 \\
= & 8 \sinh(t) \left(\sum_{n \in \mathbb{Z}} (-\Delta)^{1/2} F_n u_n^+ \right)^2 \\
= & \frac{1}{2\pi} 8 \sinh(t) \left(\int_0^{2\pi} (-\Delta)^{1/2} F u^+ d\theta \right)^2 = \frac{1}{2\pi} 8 \sinh(t) \left(\int_0^{2\pi} \partial_t F u^+ d\theta \right)^2. \quad (117)
\end{aligned}$$

Estimate of (116)

$$\begin{aligned}
& \sum_{n,m \in \mathbb{Z}} (|m|n - |n|m) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [b_m^- b_n^-] \\
&= \sum_{n>0, m<0} (-2mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [b_m^- b_n^-] \\
&\quad + \sum_{n<0, m>0} (2mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [b_m^- b_n^-] \\
&= \sum_{n>0, m<0} (-2mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [b_m^- b_n^-] \\
&\quad + \sum_{n>0, m<0} (2(-m)(-n)) [e^{-t|-n+m+1|} - e^{-t|-n+m-1|}] [b_{-m}^- b_{-n}^-] \\
&= -4 \sum_{n>0, m<0} (mn) [e^{-t|n-m+1|} - e^{-t|n-m-1|}] [b_m^- b_n^-] \\
&= 4 \sum_{n>0, m>0} (-m)n [e^{-t|n+m+1|} - e^{-t|n+m-1|}] [b_{-m}^- b_n^-] \\
&= 4 \sum_{n>0, m>0} mn [e^{-t(n+m+1)} - e^{-t(n+m-1)}] [b_m^- b_n^-] \\
&= 4 (e^{-t} - e^t) \left| \sum_{n>0} n e^{-tn} b_n^- \right|^2 = -8 \sinh(t) \left| \sum_{n \in \mathbb{Z}} i n e^{-t|n|} u_n^- \right|^2 \\
&= -8 \sinh(t) \left| \sum_{n \in \mathbb{Z}} (\partial_\theta F_n) u_n^- \right|^2 \\
&= -\frac{1}{2\pi} 8 \sinh(t) \left| \int_0^{2\pi} (\partial_\theta F) u^- d\theta \right|^2. \tag{118}
\end{aligned}$$

From (115) and (117) and (118) it follows

$$\left| \int_0^{2\pi} \partial_t F(t, \theta) u(\theta) d\theta \right| = \left| \int_0^{2\pi} \partial_\theta F(t, \theta) u(\theta) d\theta \right| \tag{119}$$

where

$$\partial_t F(t, \theta) = -2e^t \frac{e^{2t} \cos(\theta) - 2e^t + \cos(\theta)}{(e^{2t} - 2e^t \cos(\theta) + 1)^2}$$

and

$$\partial_\theta F(t, \theta) = -2e^t \frac{\sin(\theta)(e^{2t} - 1)}{(e^{2t} - 2e^t \cos(\theta) + 1)^2}.$$

3.3 Pohozaev Identities for the Laplacian in \mathbb{R}^2

In this section we derive a Pohozaev identity in $2D$ which analogous to that found in $1D$. Instead of integrating on balls, the strategy is to multiply by the fundamental solution of the heat equation. A similar idea has been preformed in [?] to study the heat flow.

The solution of

$$\begin{cases} \partial_t G + (-\Delta)G = 0 & t > 0 \\ G(0, x) = \delta_{x_0} & t = 0. \end{cases} \quad (120)$$

is given by $G(x, t) = (4\pi t)^{-1/2} e^{-\frac{|x-x_0|^2}{4t}}$.

Theorem 3.3 [Case on \mathbb{R}^2] Let $u \in \mathbb{C}^2(\mathbb{R}^2, \mathbb{R}^m)$ such that

$$\left\langle \frac{\partial u}{\partial x_i}, \Delta u \right\rangle = 0 \quad \text{a.e in } \mathbb{R}^2 \quad (121)$$

$i = 1, 2$. Assume that

$$\int_{\mathbb{R}} |u - u_0| dx < +\infty, \quad \int_{\mathbb{R}} |\nabla u(x)| dx < +\infty. \quad (122)$$

Then for all $x_0 \in \mathbb{R}^2$ and $t > 0$ the following identity holds

$$\iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 \left| \frac{\partial u}{\partial \nu} \right|^2 dx = \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \left| \frac{\partial u}{\partial \theta} \right|^2 dx. \quad (123)$$

Proof. We multiply the equation (121) by $x_i e^{-\frac{|x-x_0|^2}{4t}}$ and we integrate:

$$\begin{aligned} 0 &= \sum_{k,i=1}^2 \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (x_i - x_{0i}) \frac{\partial u}{\partial x_i} \frac{\partial^2 u}{\partial x_k^2} dx \\ &= -\frac{1}{2t} \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 \left| \frac{\partial u}{\partial \nu} \right|^2 dx - \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |\nabla u|^2 dx \\ &\quad - \frac{1}{2} \sum_{i=1}^2 \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} (x - x_0)_i \frac{\partial}{\partial x_i} |\nabla u|^2 dx \\ &= -\frac{1}{2t} \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 \left| \frac{\partial u}{\partial \nu} \right|^2 dx + \frac{1}{4t} \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 |\nabla u|^2 dx \end{aligned} \quad (124)$$

Since $\nabla u = (\frac{\partial u}{\partial \nu}, |x - x_0|^{-1} \frac{\partial u}{\partial \theta})$, from (124) we get the identity

$$\iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0|^2 \left| \frac{\partial u}{\partial \nu} \right|^2 dx = \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \left| \frac{\partial u}{\partial \theta} \right|^2 dx \quad (125)$$

and we conclude. \square

Remark 3.1 We obtain an analogous identity to (123) if we multiply the equation (121) by $X_i e^{-\frac{|x-x_0|^2}{4t}}$ where $X_1 + iX_2: \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic function. By using the Cauchy-Riemann differential equations we get

$$2 \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} |x - x_0| \left\langle \frac{\partial u}{\partial \nu}, \frac{\partial u}{\partial X} \right\rangle dx = \iint_{\mathbb{R}^2} e^{-\frac{|x-x_0|^2}{4t}} \langle x - x_0, X \rangle |\nabla u|^2 dx. \quad (126)$$

4 Compactness and Quantization of horizontal 1/2 harmonic maps

The proof of the first part of Theorem 1.2 is exactly the same of that of Lemma 2.3 in [4] and we omit it.

As far as the quantization issue is concerned the proof goes as that of Theorem 1.1 in [4] (namely the decomposition of the \mathbb{R} into converging regions, bubbles domains and neck-regions) once we perform the analysis of the neck-region. As we have already mentioned in the Introduction global uniform $L^{2,1}$ estimates in degenerating annuli are not anymore available as in the case of 1/2-harmonic maps with values into a sphere. Therefore we have to perform a subtle analysis of $(-\Delta)^{1/4}u$ where $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ is a horizontal 1/2-harmonic map in an annular domain. To this purpose we will make use of the Pohozaev-type formulae we have discovered in Section 3.1.

We first show that if $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ is a horizontal 1/2-harmonic map with $(-\Delta)^{1/2}u \in L^1(\mathbb{R})$ then $\mathcal{R}[(P^N(-\Delta)^{1/4}u) \in L^{2,1}(\mathbb{R})$.

Next we show that if $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is a weak harmonic map, then $(-\Delta)^{1/2}u \in L^1(\mathbb{R})$.

Proposition 4.1 *Let $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ be a weak harmonic maps. Then $(-\Delta)^{1/2}u \in L^1(\mathbb{R})$.*

Proof of Proposition 4.1. Step 1. We prove that $(-\Delta)^{1/2}u \in L^1(\mathbb{R})$.

This follows from the fact that for all $\xi, \eta \in \mathcal{N}$ we have

$$P^N(\xi) \cdot (\xi - \eta) = O(|\xi - \eta|^2).$$

If $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ is a weak harmonic map, then $P^N(u)(-\Delta)^{1/2}u = (-\Delta)^{1/2}u$ in $\mathcal{D}'(\mathbb{R})$. Therefore

$$\begin{aligned} \int_{\mathbb{R}} |(-\Delta)^{1/2}u(x)| dx &= \int_{\mathbb{R}} |P_N(u(x))(-\Delta)^{1/2}u(x)| dx \\ &\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy \\ &= C \|u\|_{\dot{H}^{1/2}(\mathbb{R})}^2 < +\infty. \quad \square \end{aligned} \quad (127)$$

Proposition 4.2 *If $u \in \mathfrak{H}^{1/2}(\mathbb{R})$ is a horizontal $1/2$ -harmonic map with $(-\Delta)^{1/2}u \in L^1(\mathbb{R})$ then $\mathcal{R}[(P^N(-\Delta)^{1/4}u)] \in L^{2,1}(\mathbb{R})$.*

Proof of Proposition 4.2. Since $u \in \mathfrak{H}^{1/2}(\mathbb{R})$, we have $P_N(u)\nabla u = P_N(u)\mathcal{R}(-\Delta)^{1/2}u = 0$. The result follows from the fact that $P_N(u)(-\Delta)^{1/4}u$ satisfies the following structure equation

$$(-\Delta)^{1/4}(P^N(-\Delta)^{1/4}u) = S(P^N, u) - \mathcal{R}[(-\Delta)^{1/4}P^N(\mathcal{R}(-\Delta)^{1/4}u)]. \quad (128)$$

We deduce in particular that $\mathcal{R}[(-\Delta)^{1/4}P^N(\mathcal{R}(-\Delta)^{1/4}u)] \in L^1(\mathbb{R})$.

Since $((-\Delta)^{1/4}P^N)(\mathcal{R}(-\Delta)^{1/4}u) \in L^1(\mathbb{R})$ it follows that $((-\Delta)^{1/4}P^N)(\mathcal{R}(-\Delta)^{1/4}u) \in \mathcal{H}^1$.⁽³⁾ The Hardy Space can be also characterized as the space of function in $f \in L^1$ such that $\mathcal{R}[f] \in L^1(\mathbb{R})$.

Hence (128) implies that $(-\Delta)^{1/4}\mathcal{R}[(P^N(-\Delta)^{1/4}u)] \in \mathcal{H}^1(\mathbb{R})$ and hence $\mathcal{R}[(P^N(-\Delta)^{1/4}u)] \in L^{2,1}(\mathbb{R})$. We conclude the proof. \square

Proof of Theorem 1.6. We have already remarked that the sequence

$$v_k = (P_T^k(-\Delta)^{1/4}u_k, \mathcal{R}[P_N^k(-\Delta)^{1/4}u_k])$$

satisfies a system of the form

$$(-\Delta)^{1/4}v_k = \Omega_k v_k + \Omega_k^1 v + \mathcal{Z}(Q_k, v_k) \quad (129)$$

where $v_k \in L^2(\mathbb{R})$, $Q_k \in \dot{H}^{1/2}(\mathbb{R})$, $\Omega_k \in L^2(\mathbb{R}, so(m))$, $\Omega_k^1 \in L^{2,1}(\mathbb{R})$, $g_k \in L^1(\mathbb{R})$ and $\mathcal{Z}: \dot{H}^{1/2}(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$ with

$$\|\Omega_k\|_{L^2(\mathbb{R})} + \|\Omega_k^1\|_{L^{2,1}(\mathbb{R})} + \|Q_k\|_{\dot{H}^{1/2}(\mathbb{R})} \leq C\|v_k\|_{L^2} \quad (130)$$

If $\delta > 0$ is small enough we can apply Proposition 2.1.

$$(P_T^k + \mathcal{R}P_N^k)(-\Delta)^{1/4}u_k(x) = \mathbb{1}_{A_{r_k, R_k}}[A^{-1}(x)\overrightarrow{c_{r_k}} \frac{1}{|x|^{1/2}}] + h_k(x) + g_k(x) \quad (131)$$

with $A_k^{-1} \in L^\infty \cap \dot{H}^{1/2}(\mathbb{R}, M_m(\mathbb{R}))$, $h_k \in L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1}R_k})$, (for every $\Lambda > 2$ such that $\Lambda r_k < (2\Lambda)^{-1}R_k$), $g_k \in L^2(\mathbb{R})$ with $\text{supp } g_k \subset (A_{r, R})^c$. Moreover we have

$$\overrightarrow{c_{r_k}} = O\left(\left(\log\left(\frac{R_k}{2\Lambda^2 r_k}\right)\right)^{-1/2}\right), \quad \text{as } k \rightarrow +\infty, \Lambda \rightarrow +\infty. \quad (132)$$

⁽³⁾See section 1.2 for the definition.

Therefore

$$P_T^k(-\Delta)^{1/4}u_k(x) = P_T^k\left(\frac{\mathbb{1}_{A_{r_k,R_k}}A_k^{-1}(x)\vec{c_{r_k}}}{|x|^{1/2}}\right) + P_T^kh(x) + P_T^kg(x) \quad (133)$$

$$P_N^k(-\Delta)^{1/4}u_k(x) = P_N^k\left(\mathcal{R}\left[\frac{\mathbb{1}_{A_{r_k,R_k}}A_k^{-1}(x)\vec{c_{r_k}}}{|x|^{1/2}}\right]\right) - P_N^k\mathcal{R}[h(x)] - P_N^k\mathcal{R}[g(x)] \quad (134)$$

Observe that $P_N^k\mathcal{R}g_k(x) \in L^{2,1}(A_{\Lambda r_k,(2\Lambda)^{-1}R_k})$ since $\text{supp}(g_k) \subset (A_{r_k,R_k})^c$. By combining (133) and (134) we get

$$(-\Delta)^{1/4}u_k(x) = P_T^k\left(\frac{\mathbb{1}_{A_{r_k,R_k}}A_k^{-1}(x)\vec{c_{r_k}}}{|x|^{1/2}}\right) + \bar{h}_k \quad (135)$$

with $\bar{h}_k = P_T^kh_k(x) + P_T^k(x) + P_N^k(-\Delta)^{1/4}u_k(x)$ which is in $L^{2,1}(A_{\Lambda r_k,(2\Lambda)^{-1}R_k})$.

Next we set $a_k(x) := P_T^kA_k^{-1}$ and we denote for simplicity by a_k^+ and a_k^- respectively the symmetric and antisymmetric parts of a_k . Since A_k^{-1} and P_T^k are in $H^{1/2}(\mathbb{R})$, they verify the following estimate (see [16]):

$$\left\|\frac{A_k^{-1}(x) - A_k^{-1}(-x)}{|x|^{1/2}}\right\|_{L^2} \leq C\|A_k^{-1}\|_{\dot{H}^{1/2}} \quad (136)$$

$$\left\|\frac{P_T^k(x) - P_T^k(-x)}{|x|^{1/2}}\right\|_{L^2} \leq C\|P_T^k\|_{\dot{H}^{1/2}} \quad (137)$$

$$(138)$$

By using the fact that A_k^{-1} and P_N^k, P_T^k are also in L^∞ we get that

$$\left\|\frac{a_k^-\vec{c_{r_k}}}{|x|^{1/2}}\right\|_{L^2(\mathbb{R})} \leq C|\vec{c_{r_k}}|.$$

Therefore we can write

$$(-\Delta)^{1/4}u_k(x) = \mathbb{1}_{A_{r_k,R_k}}\frac{a_k^+\vec{c_{r_k}}}{|x|^{1/2}} + \bar{h}_k + \tilde{g}_k \quad (139)$$

where $\bar{h}_k \in L^{2,1}(A_{\Lambda r_k,(2\Lambda)^{-1}R_k})$, with

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\bar{h}_k\|_{L^{2,1}(B(0,(2\Lambda)^{-1}R_k) \setminus B(0,\Lambda r_k))} < +\infty,$$

and $\tilde{g}_k = \mathbb{1}_{A_{r_k,R_k}}\frac{a_k^-\vec{c_{r_k}}}{|x|^{1/2}} \in L^2(\mathbb{R})$, with $\|\tilde{g}_k\|_{L^2} \leq |\vec{c_{r_k}}|$. We can conclude. \square

Proof of Theorem 1.7

Under the current hypotheses Theorem 1.6 and Lemma 2.2 it follows :

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow \infty} \|(-\Delta)^{1/4} u_k\|_{L^{2,\infty}(B(0,(2\Lambda)^{-1}R_k) \setminus B(0,\Lambda r_k))} = 0 \quad (140)$$

and

$$(-\Delta)^{1/4} u_k(x) = \mathbb{1}_{A_{r_k,R_k}} \frac{a_k^+ \vec{c}_{r_k}}{|x|^{1/2}} + \bar{h}_k + \tilde{g}_k, \quad (141)$$

where $\bar{h}_k \in L^{2,1}(A_{\Lambda r_k,(2\Lambda)^{-1}R_k})$, with

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow \infty} \|\bar{h}_k\|_{L^{2,1}(B(0,(2\Lambda)^{-1}R_k) \setminus B(0,\Lambda r_k))} < +\infty,$$

$\tilde{g}_k \in L^2(\mathbb{R})$ and $\|\tilde{g}_k\|_{L^2} \leq C|\vec{c}_r|$. By combining (140) and (141) we get

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow \infty} \|((-\Delta)^{1/4} u_k)^-\|_{L^2(B(0,(2\Lambda)^{-1}R_k) \setminus B(0,\Lambda r_k))} = 0 \quad (142)$$

In order to establish a link between the symmetric and antisymmetric part of $(-\Delta)^{1/4} u_k$ we make use of the formula 87 than can be rewritten as

$$\left(\int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\sin(\arctan(-x))}{(1+x^2)^{3/4}} \right) w(tx) dx \right)^2 = \left(\int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right) w(tx) dx \right)^2 \quad (143)$$

Now we plug into (143) the function $w_k(x) = \mathbb{1}_{A_{\Lambda r_k,(2\Lambda)^{-1}R_k}} (-\Delta)^{1/4} u_k(x)$. We observe that

$$\int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\sin(\arctan(-x))}{(1+x^2)^{3/4}} \right) \mathbb{1}_{A_{r_k,R_k}}(tx) \frac{a_k^+ \vec{c}_{r_k}}{|tx|^{1/2}} dx = 0.$$

Therefore we have that

$$\begin{aligned} & \int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right) \mathbb{1}_{A_{r_k,R_k}}(tx) \frac{a_k^+ \vec{c}_{r_k}}{|tx|^{1/2}} dx \\ &= \int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\sin(\arctan(-x))}{(1+x^2)^{3/4}} \right) \tilde{w}_k(tx) dx - \int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right) \tilde{w}_k(tx) dx. \end{aligned}$$

where

$$\tilde{w}_k(x) = w_k(x) - \mathbb{1}_{A_{r_k,R_k}}(x) \frac{a_k^+ \vec{c}_{r_k}}{|x|^{1/2}} = \bar{h}_k + \tilde{g}_k.$$

Next we use the fact that the operators

$$M^-[w](t) := \int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\sin(\arctan(-x))}{(1+x^2)^{3/4}} \right) w(tx) dx, \quad (144)$$

$$M^+[w](t) := \int_{\mathbb{R}} \sqrt{\pi} \left(\frac{\cos(\arctan(-x))}{(1+x^2)^{3/4}} \right) w(tx) dx, \quad (145)$$

are isomorphism from L_+^p to L_+^p (resp. L_-^p to L_-^p) for every $p > 1$ and from $L_+^{2,1}$ to $L_+^{2,1}$ (resp. $L_-^{2,1}$ to $L_-^{2,1}$) and we deduce that

$$\mathbb{1}_{A_{r_k, R_k}}(x) \frac{a_k^+ \overrightarrow{c_{r_k}}}{|x|^{1/2}} = \varphi_k + \psi_k.$$

with $\varphi_k \in L^{2,1}(A_{\Lambda r_k, (2\Lambda)^{-1} R_k})$, $\psi_k \in L^2(\mathbb{R})$ and $\|\psi_k\|_{L^2} \leq C|\overrightarrow{c_r}|$.

Hence

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|(-\Delta)^{1/4} u_k - \psi_k\|_{L^2(B(0, (2\Lambda)^{-1} R_k) \setminus B(0, \Lambda_k r_r))} \quad (146)$$

$$\begin{aligned} &\leq \limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|(-\Delta)^{1/4} u_k - \psi_k\|_{L^{2,\infty}(B(0, (2\Lambda)^{-1} R_k) \setminus B(0, \Lambda_k r_r))} \\ &\cdot \limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|(-\Delta)^{1/4} u_k - \psi_k\|_{L^{2,1}(B(0, (2\Lambda)^{-1} R_k) \setminus B(0, \Lambda_k r_r))} = 0. \end{aligned} \quad (147)$$

Therefore since $\limsup_{k \rightarrow +\infty} \|\psi_k\|_{L^2} = 0$ we deduce that

$$\limsup_{\Lambda \rightarrow \infty} \limsup_{k \rightarrow +\infty} \|(-\Delta)^{1/4} u_k\|_{L^2(B(0, (2\Lambda)^{-1} R_k) \setminus B(0, \Lambda_k r_r))}.$$

We can conclude the proof of Theorem 1.7. □

5 Counter-example

The aim of this section is to construct a sequence of solutions of a Schrödinger type equation with antisymmetric potential whose energy is not quantized. In particular, we are going to build a sequence of solutions whose energy of the potential goes to zero in the neck but not the energy of the solutions.

We consider $u(t) = 1$ on $[-1, 1]$ and $\frac{1}{|t|^{1/2}}$ elsewhere and $v(t) = \frac{1}{(1+t^2)^{3/8}}$.

Lemma 5.1 *As $t \rightarrow +\infty$, we have*

$$(-\Delta)^{1/4} u(t) = O\left(\frac{1}{t^{3/2}}\right)$$

and

$$(-\Delta)^{1/4} v(t) = O\left(\frac{1}{t^{5/4}}\right).$$

Proof :

$$\begin{aligned}
(-\Delta)^{\frac{1}{4}}u(t) &= \int_{\mathbb{R}} \frac{u(t) - u(s)}{|t - s|^{\frac{3}{2}}} ds \\
&= \int_{-\infty}^{-1} \frac{u(t) - u(s)}{|t - s|^{\frac{3}{2}}} ds + \int_{-1}^1 \frac{u(t) - u(s)}{|t - s|^{\frac{3}{2}}} ds + \int_1^{+\infty} \frac{u(t) - u(s)}{|t - s|^{\frac{3}{2}}} ds \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{148}$$

Let $t > 1$

$$\begin{aligned}
I_2 &= \int_{-1}^1 \frac{t^{-\frac{1}{2}} - 1}{|t - s|^{\frac{3}{2}}} ds \\
&= \frac{t^{-\frac{1}{2}} - 1}{t^{\frac{3}{2}}} \int_{-1}^1 \frac{1}{|1 - \frac{s}{t}|^{\frac{3}{2}}} ds \\
&= \frac{t^{-\frac{1}{2}} - 1}{t^{\frac{1}{2}}} \int_{-\frac{1}{t}}^{\frac{1}{t}} \frac{1}{|1 - u|^{\frac{3}{2}}} du \\
&= \frac{t^{-\frac{1}{2}} - 1}{t^{\frac{1}{2}}} \left[2(1 - u)^{-\frac{1}{2}} \right]_{-\frac{1}{t}}^{\frac{1}{t}} = O\left(\frac{1}{t^{\frac{3}{2}}}\right)
\end{aligned} \tag{149}$$

$$\begin{aligned}
I_1 &= \int_{-\infty}^{-1} \frac{t^{-\frac{1}{2}} - (-s)^{-\frac{1}{2}}}{|t - s|^{\frac{3}{2}}} ds \\
&= \frac{1}{t^2} \int_{-\infty}^{-1} \frac{1 - (-\frac{s}{t})^{-\frac{1}{2}}}{|1 - \frac{s}{t}|^{\frac{3}{2}}} ds \\
&= \frac{1}{t} \int_{\frac{1}{t}}^{+\infty} \frac{1 - u^{-\frac{1}{2}}}{(1 + u)^{\frac{3}{2}}} du
\end{aligned} \tag{150}$$

$$\begin{aligned}
I_3 &= \int_1^{+\infty} \frac{t^{-\frac{1}{2}} - s^{-\frac{1}{2}}}{|t - s|^{\frac{3}{2}}} ds \\
&= \frac{1}{t^2} \int_1^{+\infty} \frac{1 - (\frac{s}{t})^{-\frac{1}{2}}}{|1 - \frac{s}{t}|^{\frac{3}{2}}} ds \\
&= \frac{1}{t} \int_{\frac{1}{t}}^{+\infty} \frac{1 - u^{-\frac{1}{2}}}{|1 - u|^{\frac{3}{2}}} du
\end{aligned} \tag{151}$$

We easily check that $\frac{1 - u^{-\frac{1}{2}}}{|1 - u|^{\frac{3}{2}}} \underset{u \rightarrow 1}{\sim} (1 - u)^{-\frac{1}{2}}$ and then the last integral is well defined.

But changing the variable u into $\frac{1}{v}$ into I_1 and I_2 we observe that

$$\int_0^{+\infty} \frac{1 - u^{-\frac{1}{2}}}{(1+u)^{\frac{3}{2}}} du = \int_0^{+\infty} \frac{1 - u^{-\frac{1}{2}}}{|1-u|^{\frac{3}{2}}} du = 0$$

Which implies that

$$I_1 = \frac{-1}{t} \int_0^{\frac{1}{t}} \frac{1 - u^{-\frac{1}{2}}}{(1+u)^{\frac{3}{2}}} du = O_{t \rightarrow +\infty} \left(\frac{1}{t^{\frac{3}{2}}} \right)$$

and

$$I_3 = \frac{-1}{t} \int_0^{\frac{1}{t}} \frac{1 - u^{-\frac{1}{2}}}{|1-u|^{\frac{3}{2}}} du = O_{t \rightarrow +\infty} \left(\frac{1}{t^{\frac{3}{2}}} \right).$$

Which proves the lemma for u when $t > 1$. Of course we have the same result for $t < -1$ by symmetry.

$$\begin{aligned} (-\Delta)^{\frac{1}{4}} v(t) &= \int_{\mathbb{R}} \frac{u(t) - u(s)}{|t-s|^{\frac{3}{2}}} ds = \frac{1}{(1+t^2)^{\frac{3}{8}}} \int_{\mathbb{R}} \frac{(1+s^2)^{\frac{3}{8}} - (1+t^2)^{\frac{3}{8}}}{(1+s^2)^{\frac{3}{8}} |t-s|^{\frac{3}{2}}} ds \\ &= \frac{1}{(1+t^2)^{\frac{3}{8}}} \int_{-\infty}^{-1} \frac{(1+s^2)^{\frac{3}{8}} - (1+t^2)^{\frac{3}{8}}}{(1+s^2)^{\frac{3}{8}} |t-s|^{\frac{3}{2}}} ds + \frac{1}{(1+t^2)^{\frac{3}{8}}} \int_{-1}^1 \frac{(1+s^2)^{\frac{3}{8}} - (1+t^2)^{\frac{3}{8}}}{(1+s^2)^{\frac{3}{8}} |t-s|^{\frac{3}{2}}} ds \\ &\quad + \frac{1}{(1+t^2)^{\frac{3}{8}}} \int_1^{+\infty} \frac{(1+s^2)^{\frac{3}{8}} - (1+t^2)^{\frac{3}{8}}}{(1+s^2)^{\frac{3}{8}} |t-s|^{\frac{3}{2}}} ds \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{152}$$

We remark that

$$\frac{(1+s^2)^{\frac{3}{8}} - (1+t^2)^{\frac{3}{8}}}{|t-s|^{\frac{3}{4}}} \text{ is uniformly bounded.}$$

Then, let $t > 1$,

$$\begin{aligned} I_2 &= O \left(\frac{1}{|t|^{\frac{3}{4}}} \int_{-1}^1 \frac{1}{(1+s^2)^{\frac{3}{8}} |t-s|^{\frac{3}{4}}} ds \right) \\ &= O \left(\frac{1}{|t|^{\frac{3}{4}}} \int_{-1}^1 \frac{1}{|t-s|^{\frac{3}{4}}} ds \right) \\ &= O \left(\frac{1}{|t|^{\frac{5}{4}}} \right) \end{aligned} \tag{153}$$

And

$$\begin{aligned}
I_1 &= O \left(\frac{1}{|t|^{\frac{3}{4}}} \int_{-\infty}^{-1} \frac{1}{(1+s^2)^{\frac{3}{8}} |t-s|^{\frac{3}{4}}} ds \right) \\
&= O \left(\frac{1}{|t|^{\frac{3}{4}}} \int_{-\infty}^{-1} \frac{1}{(-s)^{\frac{3}{4}} |t-s|^{\frac{3}{4}}} ds \right) \\
&= O \left(\frac{1}{|t|^{\frac{3}{4}}} \int_{-\infty}^{-1} \frac{1}{(-s)^{\frac{3}{4}} |t-s|^{\frac{3}{4}}} ds \right) \\
&= O \left(\frac{1}{|t|^{\frac{5}{4}}} \int_{-\infty}^{-1/t} \frac{1}{(-u)^{\frac{3}{4}} |1-u|^{\frac{3}{4}}} du \right) \\
&= O \left(\frac{1}{|t|^{\frac{5}{4}}} \right)
\end{aligned} \tag{154}$$

The same estimate work for I_3 and $t < -1$, which achieved the proof. \square

Then we set $\omega = \frac{(-\Delta)^{\frac{1}{4}} u}{v}$ and $\omega_1 = \frac{(-\Delta)^{\frac{1}{4}} v + \omega u}{u}$, which gives

$$(-\Delta)^{\frac{1}{4}} U = \Omega U + \Omega_1 U$$

with

$$U = \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

and

$$\Omega_1 = \begin{pmatrix} 0 & 0 \\ \omega_1 & 0 \end{pmatrix}.$$

Finally, we set $U_n(t) = c_n U(nt)$, $\Omega_n = \sqrt{n} \Omega(nt)$, and $\Omega_{1n} = \sqrt{n} \Omega_1(nt)$. Where

$$c_n = \frac{1}{\|u(nt)\|_2} = \left(\frac{n}{\ln \left(\frac{n+\sqrt{1+n^2}}{-n+\sqrt{1+n^2}} \right)} \right)^{\frac{1}{2}} \sim \left(\frac{n}{\ln(n)} \right)^{\frac{1}{2}}$$

We easily check that

$$\|U_n\|_2 \sim 1$$

$$\lim_{R \rightarrow +\infty} \lim_{n \rightarrow +\infty} \|\Omega_n\|_{L^2(B(0,1/R) \setminus B(0,R/n))} + \|\Omega_{1n}\|_{L^{2,1}(B(0,1/R) \setminus B(0,R/n))} = 0$$

and

$$(-\Delta)^{\frac{1}{4}}U_n = \Omega_n U_n + \Omega_{1n} U_n$$

which prove that there is no quantification of the energy, since Ω_n and Ω_{1n} have no energy in the neck region despite U_n get some. Of course such an example doesn't satisfies a Pohozaev identity, since it is symmetric. It is a "local" example since U_n bounded in L^2 only on $[-1, 1]$, it would be interesting to construct a example define on the whole \mathbb{R} . It may be possible by starting on the circle and then projecting on to \mathbb{R} . Indeed the main idea here is to take a modification of the Green function of the $(-\Delta)^{\frac{1}{4}}$. Hence playing this game on S^1 and projecting then \mathbb{R} should provide an example with good decreasing at infinity.

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